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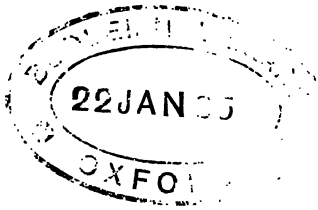

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NEW-STREET SQUARE

PREFACE.

THERE are few persons engaged in tuition who have not experienced the difficulty of teaching Euclid to young pupils, more especially to those who have not acquired habits of close application, or who have no natural aptitude for mathematical studies. One reason for this arises probably from the nature of the subject itself, which requires a more sustained attention, and a greater concentration of thought and power of abstraction, than beginners can ordinarily be induced to apply to *any* subject at the age when the study of Euclid is commonly begun in schools. But another, and undoubtedly the chief cause of the difficulty, is the absence in the ordinary editions of the Elements of those aids to the learner which are so plentifully supplied in every other department of instruction.

Such assistance it is the special design of the present edition to afford, partly by the use of a peculiar type in those parts of a proposition which require to be distinguished from each other, and partly by a new arrangement of the figures and demonstrations, which it is hoped will be found to embody some important improvements on the ordinary method of presenting the subject to beginners. The chief features of this arrangement may be summed up as follows: —

1. Immediately following the enunciation are given, in each case, the 'references,' or elements, of the proposition — the definitions, axioms, and previous propositions — on which the successive steps of the reasoning depend. These must be thoroughly mastered by the pupil; and they should in all cases be required to be written out, or repeated aloud, before either

the construction or the proof is attempted. Every teacher knows how carelessly these references, generally inserted in text-books in the margin, or in the body of the proposition, are slurred over, or repeated mechanically, without any reference to their import, by boys in class, and how confused and imperfect is their conception of the reasoning in consequence.

2. In describing the figures, the parts which are given in the enunciation are represented by dark lines, and those which are added in the course of the demonstration by dotted lines. The process of the construction is thus exhibited to the eye, and the *data* and the *quæsita* of the problem can always be distinguished at a glance.

3. In the demonstrations, the several steps of the proof are arranged in a logical form, by giving the premisses and the conclusion always in separate lines, and in a different type; and, as a further aid to the learner, the enunciations are broken into paragraphs, and the demonstration into corresponding divisions, wherever the proposition consists of more than one case.

In this way the constituent parts of a proposition are presented separately, part by part, and the learner, knowing exactly where one begins and the other ends, is enabled to make himself master of the one before he proceeds to the other.

The symbolical editions of Hill, Blakelock, and Williams, were among the earliest to show the advantages of printing separately the parts of a proposition and its demonstration; and they have been followed with great success by Mr. Potts and other recent writers. The plan adopted by them of printing every sentence, or part of a sentence, which contains a new step in the reasoning, in a *separate line*, has been followed in the present edition, but to a greater extent, and with increased distinctness. To avoid the confused appearance produced by the lines being scattered irregularly over the page, as in Mr. Potts's and some other editions, the lines have been printed so as to commence uniformly from the side of the page. Every *conclusion is what* printers technically term 'indented,' and *the applicable part of* it — that is, the part made use of or

referred to in the subsequent reasoning — is, to mark its importance, printed in italics.

The conclusions thus distinguished, or the most important of them, if entered in the 'Copy-books' prepared for this purpose, and published in connexion with this work, will supply a kind of *Analysis of Euclid*, to those who have gone through the subject, but who wish at any time, as on the approach of an examination, to refresh their memory by a cursory re-perusal. To such, the reading of the several steps, and an inspection of the figure, will, in most cases, be sufficient to recall the complete proof to the mind, without the trouble of going over the entire proposition a second time; and this will be a pleasing and most improving exercise, and tend strongly to impress not only the proof itself, but also the principle of the proof, on the memory.

Although the text of Dr. Simson has been, in the main, adhered to in the present edition, alterations have been made wherever there appeared to be any obscurity in the language which could be removed by the introduction of a step, or the variation or transposition of a sentence.

As examples of such alteration, may be mentioned, the introduction of an additional figure in prop. 27, book i., and the use of a definite form of expression to mark the distinction in indirect demonstrations between a conclusion true in itself and one correctly deduced, but from an incorrect hypothesis. In the former case, the conclusion is expressed in the ordinary way; in the latter, the word 'assumed' is employed in the premiss, and the word 'must' in the conclusion, to indicate that the reasoning proceeds on a false assumption, although the reasoning itself is correct. This distinction, it is believed, will be found of considerable practical importance in teaching Euclid to young students.

A. K. Isbister.

Stationers' School:
August 1, 1862.

THE SCHOOL EUCLID.

BOOK I.

DEFINITIONS.

I.

A POINT is that which hath no parts, or which hath no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which hath only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

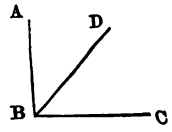
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

'A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.'

IX.

A plane rectilineal angle is the inclination of two straight lines to one another, which meet *together, but* are not in *the same straight line.*



B

N.B.—'When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines containing the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line: Thus the angle which is contained by the straight lines, AB, CB, is named the angle ABC, or CBA; that which is contained by AB, DB, is named the angle ABD, or DBA; and that which is contained by DB, CB, is called the angle DBC, or CBD; but, if there be only one angle at a point, it may be expressed by a letter placed at that point: as the angle at E.'

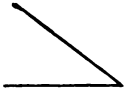
X.

When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

XI.

An obtuse angle is that which is greater than a right angle.

XII.



An acute angle is that which is less than a right angle.

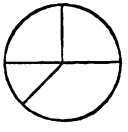
XIII.

'A term or boundary is the extremity of any thing.'

XIV.

A figure is that which is enclosed by one or more boundaries.

XV.

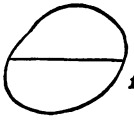


A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

XVI.

And this point is called the centre of the circle.

XVII.



A diameter of a circle is a straight line drawn through *the centre*, and terminated both ways by the circum*ference*.

XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

'A segment of a circle is the figure contained by a straight line, and the circumference it cuts off.'

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines.

XXII.

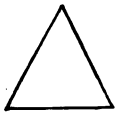
Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by more than four straight lines

XXIV.

Of three-sided figures, an equilateral triangle is that which has three equal sides.



XXV.

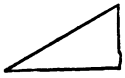
An isosceles triangle, is that which has only two sides equal.

XXVI.

A scalene triangle, is that which has three unequal sides.

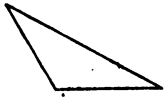
XXVII.

A right-angled triangle, is that which has a right angle



XXVIII.

An obtuse-angled triangle, is that which has an obtuse angle.



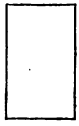
XXIX.

An acute-angled triangle, is that which has three acute angles.

XXX.

Of four-sided figures, a square is that which has all its sides equal, and all its angles right angles.

XXXI.



An oblong, is that which has all its angles right angles, but has not all its sides equal.

XXXII.

A rhombus, is that which has all its sides equal, but its angles are not right angles.

XXXIII.

A rhomboid, is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

XXXIV.

All other four-sided figures besides these, are called Trapeziums.

XXXV.

Parallel straight lines, are such as are in the same plane, and which being produced ever so far both ways do not meet.

POSTULATES.

I.

Let it be granted that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any centre, at any distance *from that centre.*

AXIOMS.

I.

Things which are equal to the same thing, are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot inclose a space.

XI.

All right angles are equal to one another.

XII.

'If a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.'

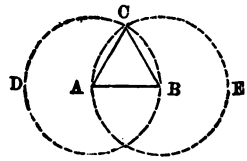
PROP. I.—Problem.

To describe an equilateral triangle upon a given finite straight line.

*(References—Def. 15; ax. 1; post. 1, 3.)

Let AB be the given straight line.

It is required to describe on AB an equilateral triangle.



CONSTRUCTION

From the centre A, at the distance AB, describe the circle BCD; (post. 3)

from the centre B, at the distance BA, describe the circle ACE;

and from the point C, in which the circles cut one another, draw the straight lines CA, CB, to the points A, B. (post. 1.)

Then ABC shall be an equilateral triangle.

DEMONSTRATION

Because the point A is the centre of the circle BCD,

therefore AC *is equal to* AB; (def. 15)

and because the point B is the centre of the circle ACE,

therefore BC *is equal to* BA.

But it has been proved that CA is equal to AB;

therefore CA, CB *are each of them equal to* AB;

but things which are equal to the same thing are equal to one another; (ax. 1)

therefore CA *is equal to* CB;

wherefore CA, AB, BC, are equal to one another;

and the triangle ABC is therefore equilateral;

and it is described upon the given straight line AB.

Which was required to be done.

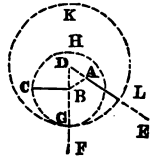
PROP. II.—Problem.

From a given point to draw a straight line equal to a given straight line.

(References—Prop. I. 1; post. 1, 2, 3; ax. 1, 3; def. 15.)

* To be written out, or repeated to the Teacher.

Let A be the given point, and BC the given straight line.
It is required to draw from A a straight line equal to BC.



CONSTRUCTION

From the point A to B draw the straight line AB; (post. 1.)
upon AB describe the equilateral triangle DAB, (1. 1)
and produce the straight lines DA, DB, to E and F;
from the centre B, at the distance BC, describe the circle CGH, (post. 3)
and from the centre D, at the distance DG, describe the circle GKL.

Then AL shall be equal to BC.

DEMONSTRATION

Because the point B is the centre of the circle CGH,
therefore BC *is equal to* BG; (*def.* 15)
and because D is the centre of the circle GKL
therefore DL *is equal to* DG,
and DA, DB, parts of them, are equal;
therefore the remainder AL *is equal to the remainder* BG. (ax. 3.)
But it has been shown that BC is equal to BG;
wherefore AL *and* BC *are each of them equal to* BG;
and things which are equal to the same thing are equal to one another;
therefore the straight line AL is equal to BC. (ax. 1.)
Wherefore from the given point A a straight line AL has been drawn equal to the given straight line BC.

Which was to be done.

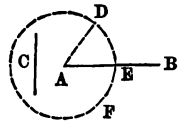
PROP. III.—PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

(References—Prop. 1. 2; def. 15; post. 3; ax. 1.)

Let B *and* C *be the* given straight lines, of which AB is greater *than* C.

It is required to cut off from AB the greater, a part equal to C, the less.



CONSTRUCTION

From the point A draw the straight line AD equal to C; (I. 2)
and from the centre A, at the distance AD, describe the circle DEF. (post. 3.)

Then AE shall be equal to C.

DEMONSTRATION

Because A is the centre of the circle DEF,

therefore AE *is equal to* AD; (def. 15)

but the straight line C is likewise equal to AD; (constr.)

therefore AE *and* C *are each of them equal to* AD;

wherefore the straight line AE is equal to C. (ax. 1.)

And therefore from AB, the greater of two straight lines, a part AE has been cut off equal to C, the less.

Which was to be done.

PROP. IV.—THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by these sides equal to each other:

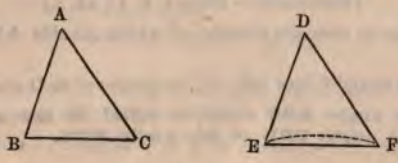
then they shall likewise have their bases, or third sides, equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.

(References—Ax. 8, 10.)

Let ABC, DEF, be two triangles, which have the two sides AB, AC, equal to the two sides, DE, DF, each to each, viz. AB to DE, and AC *to DF;*

and the angle BAC equal to the angle EDF.

Then the base BC shall be equal to the base EF;
and the triangle ABC to the triangle DEF,
and the other angles to which the equal sides are opposite, shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



DEMONSTRATION

For, if the triangle ABC be applied to the triangle DEF,
so that the point A may be on D, and the straight line AB upon DE;
because AB is equal to DE,

therefore the point B *shall coincide with the point* E;

and AB coinciding with DE,
and the angle BAC being equal to the angle EDF, (hyp.)

therefore AC *shall coincide with* DF;

and because AC is equal to DF,

wherefore also the point C *shall coincide with the point* F,

But the point B coincides with the point E;

wherefore the base BC *shall coincide with the base* EF;

because the point B coinciding with E, and C with F,
if the base BC do *not* coincide with the base EF,
two straight lines would inclose a space, which is impossible. (ax. 10.)

Therefore the base BC shall coincide with the base EF, and be equal to it.

Wherefore the whole triangle ABC shall coincide with the whole triangle DEF, and be equal to it;

and the other angles of the one shall coincide with the remaining angles of the other, and be equal to them, viz.:

the angle ABC to the angle DEF,
and the angle ACB to the angle DFE.

Therefore, if two triangles have two sides of the one equal to two sides of the other, &c.

Which was to be demonstrated.

PROP. V.—THEOREM.

The angles at the base of an isosceles triangle are equal to one another;

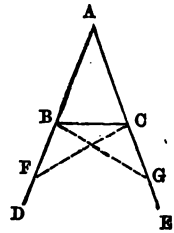
and if the equal sides be produced, the angles upon the other side of the base shall be equal.

(References — Prop. I. 3, 4; AX. 3.)

Let ABC be an isosceles triangle, of which the side AB is equal to AC,

and let the straight lines AB, AC, be produced to D and E.

Then the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.



CONSTRUCTION

In BD take any point F,

and from AE, the greater, cut off AG equal to AF, the less, (I. 3) and join FC, GB.

DEMONSTRATION

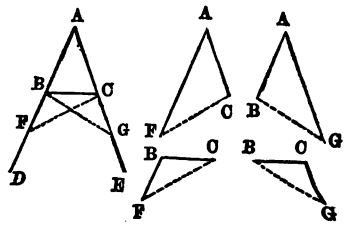
Because AF is equal to AG; (constr.)

and AB to AC; (hypoth.)

the two sides FA, AC, are equal to the two GA, AB, each to each,

and they contain the angle FAG common to the two triangles AFC, AGB;

To assist the learner, the figure may be drawn so as to exhibit to the eye the triangles of which it is composed, in the following manner:—



therefore the base FC *is equal to the base* GB,

and the triangle AFC to the triangle AGB;

and the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite, viz.

the angle ACF *to the angle* ABG,

and the angle AFC *to the angle* ABG. (I. 4.)

And because the whole AF, is equal to the whole AG,

of which the parts AB, AC, are equal;

the remainder BF *shall be equal to the remainder* CG; (ax. 3.)

and FC was proved to be equal to GB,

therefore the two sides BF, FC, are equal to the two CG, GB, each to each;

and the angle BFC is equal to the angle CGB,

and the base BC is common to the two triangles BFC, CGB; wherefore these triangles are equal, and their remaining angles, each to each, to which the equal sides are opposite;

therefore the angle FBC *is equal to the angle* GCB;

and the angle FBC *is equal to the angle* CBG. (I. 4.)

And, since it has been demonstrated,

that the whole angle ABG is equal to the whole angle ACF,

the parts of which, the angles CBG, BCF, are equal;

therefore the remaining angle ABC is equal to the remaining angle ACB,

which are the angles at the base of the triangle ABC.

And it has been proved,

that the angle FBC is equal to the angle GCB,

which are the angles upon the other side of the base.

Therefore the angles at the base, &c. Q.E.D.

Cor. Hence every equilateral triangle is also equiangular.

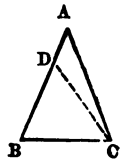
PROP. VI.—THEOREM.

If two angles of a triangle be equal to each other;

then the sides also which subtend, or are opposite to, the equal angles *shall be equal to one another.*

(References — Prop. I. 3, 4.)

Let the triangle ABC have the angle ABC equal to the angle ACB.
Then the side AB shall be equal to the side AC.



CONSTRUCTION

For, if AB be not equal to AC,
one of them is greater than the other.
Let AB be the greater;
and from it cut off DB equal to AC, the less, (I. 3.) and join DC.

DEMONSTRATION

Then in the triangles DBC, ACB,
because DB is assumed to be equal to AC, and BC is common to both,
the two sides DB, BC, are equal to the two AC, CB, each to each;
and the angle DBC is equal to the angle ACB; (hyp.)
therefore the base DC *must be equal to the base* AB,
and the triangle DBC *to the triangle* ACB, (I. 4)
the less equal to the greater, which is absurd.
Therefore AB *is not unequal to* AC,
that is AB is equal to AC.
Wherefore, if two angles, &c. Q.E.D.

Cor. Hence every equiangular triangle is also equilateral.

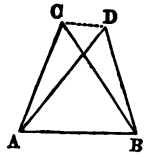
PROP. VII.—THEOREM.

Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to each other, and likewise those which are terminated in the other extremity.

(References — Prop. I. 5; ax. 9.)

If it be possible, let there be two triangles, ACB, ADB, upon the *same base* AB, and upon the same side of it, which have their sides *CA, DA, terminated* in the extremity A of the base, equal to one *another, and likewise* their sides CB, DB, that are terminated in B

First. Let the vertex of each of the triangles be without the other.



CONSTRUCTION

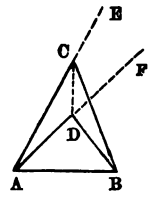
Join the vertices C and D by the straight line CD.

DEMONSTRATION

Because, in the triangle ACD, AC is assumed to be equal to AD,
therefore the angle ACD *must be equal to the angle* ADC; (I. 5)
but the angle ACD is greater than the angle BCD; (AX. 9)
therefore the angle ADC must be greater also than BCD;
much more then must the angle BDC *be greater than the angle* BCD.

Again, because in the triangle BCD, CB is assumed to be equal to BD,
therefore the angle BDC *must be equal to the angle* BCD; (I. 5)
but the angle BDC has been proved to be greater than the angle BCD;
therefore the angle BDC *must be both equal to, and greater than the angle* BCD;
which is impossible.

Secondly. Let the vertex D of the triangle ADB fall within the triangle ACB.



CONSTRUCTION

Join the vertices C and D, and produce AC, AD to E, F.

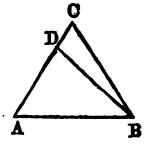
DEMONSTRATION

Because in the triangle ACD, AC is assumed to be equal to AD,
therefore the angles ECD, FDC *upon the other side of the base* CD, *must be equal to one another*; (I. 5)

but the angle ECD is greater than the angle BCD ; (ax. 9)
wherefore the angle FDC must likewise be greater than the angle BCD ;
much more then must the angle BDC *be greater than the angle* BCD.

Again, because in the triangle BCD, CB is assumed to be equal to DB,
therefore the angle BDC *must be equal to the angle* BCD ; (I. 5)
but BDC has been proved to be greater than BCD ;
wherefore the angle BDC *must be both equal to, and greater than, the angle* BCD ;
which is impossible.

Thirdly. Let the vertex of one triangle be upon a side of the other.



This case needs no demonstration, for it is evident from inspection that the side AC is greater than its part AD.
Therefore upon the same base and on the same side of it, &c.
Q. E. D.

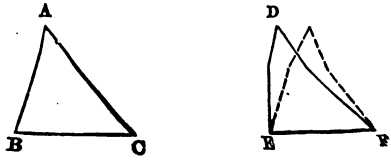
PROP. VIII.—THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal;
then the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

(References—Prop. I. 7 ; ax. 8.)

Let ABC, DEF be two triangles, having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF ; and also the base BC equal to the base EF.

Then the angle BAC shall be equal to the angle EDF.



DEMONSTRATION

For, if the triangle ABC be applied to DEF,
so that the point B be on E, and the straight line BC upon EF;
then because BC is equal to EF, (hyp.)

therefore the point C *shall coincide with the point* F;

wherefore BC coinciding with EF,

BA *and* AC *shall coincide with* ED *and* DF;

for if the base BC coincides with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG,

then, upon the same base EF, and upon the same side of it, there can be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise their sides which are terminated in the other extremity;

but this is impossible. (I. 7.)

Therefore, if the base BC coincides with the base EF,

the sides BA, AC, *cannot but coincide with the sides* ED, DF;

wherefore the angle BAC coincides with the angle EDF, and is equal to it. (ax. 8.)

Therefore, if two triangles, &c. Q. E. D.

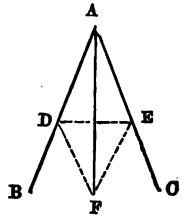
PROP. IX.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

(References — Prop. I. 1, 3, 8.)

Let the angle BAC be the given rectilineal angle.

It is required to bisect it.



CONSTRUCTION

Take any point D in AB,

from AC cut off AE equal to AD, (I. 3) and join DE;

upon the side of it opposite to A, describe the equilateral triangle DEF, (I. 1) and join AF.

Then the straight line AF bisects the angle BAC.

DEMONSTRATION

Because AD is equal to AE, (constr.)
and AF is common to the two triangles DAF, EAF;
the two sides DA, AF, are equal to the two sides EA, AF, each to each;
and the base DF is equal to the base EF; (constr.)

therefore the angle DAF *is equal to the angle* EAF. (I. 8.)

Wherefore the given rectilineal angle BAC is bisected by the line AF. Q. E. F

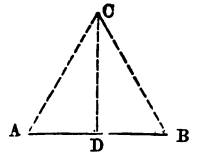
PROP. X. — PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.

(References — Prop. I. 1, 4, 9.)

Let AB be the given straight line.

It is required to divide AB into two equal parts.



CONSTRUCTION

Upon the straight line AB describe the equilateral triangle ABC, (I. 1) and bisect the angle ACB by the straight line CD. (I. 9.)

Then AB shall be divided into two equal parts in the point D.

DEMONSTRATION

Because AC is equal to CB, (constr.)
and CD common to the two triangles ACD, BCD,
the two sides AC, CD, are equal to the two sides BC, CD, each to each;
and the angle ACD is equal to the angle BCD; (constr.)

therefore the base AD *is equal to the base* DB. (I. 4.)

Wherefore the straight line AB is divided into two equal parts in the point D. Q. E. F.

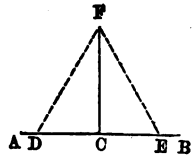
PROP. XI. — Problem.

To draw a straight line at right angles to a given straight line, from a given point in the same.

(References — Prop. I. 1, 3, 8; ax. 1; def. 10.)

Let AB be the given straight line, and C the given point in it.

It is required to draw a straight line from the point C at right angles to AB.



CONSTRUCTION

In AC, take any point D,
and make CE equal to CD; (I. 3)
upon DE describe the equilateral triangle DFE, (I. 1)
and join FC.

Then FC, drawn from the point C, shall be at right angles to AB.

DEMONSTRATION

Because DC is equal to CE,
and CF is common to the two triangles DCF, ECF;
the two sides DC, CF, are equal to the two EC, CF, each to each;
and the base DF is equal to the base EF; (constr.)

therefore the angle DCF *is equal to the angle* ECF; (I. 8)

and they are adjacent angles.

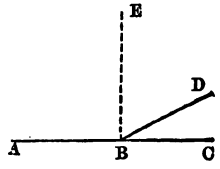
But, 'when the adjacent angles which one straight line makes with another straight line, are equal to one another, each of them is called a right angle;' (def. 10)

therefore each of the angles DCF, ECF, *is a right angle.*

Wherefore, from the point C, in the straight line AB, FC has been drawn at right angles to AB.

Q. E. F.

Cor. By help of this problem, it may be demonstrated that two right lines cannot have a common segment.



If it be possible, let the two straight lines ABC, ABD, have the segment AB common to both of them.

CONSTRUCTION

From the point B, draw BE at right angles to AB. (I. 11.)

DEMONSTRATION

Because ABC is a straight line,

therefore the angle CBE *is equal to the angle* EBA; (def. 10)

in the same manner, because ABD is assumed to be a straight line,

therefore the angle DBE *must be equal to the angle* EBA;

wherefore the angle DBE *must be equal to the angle* CBE, (ax. 1)

the less to the greater, which is impossible.

Therefore two straight lines cannot have a common segment.

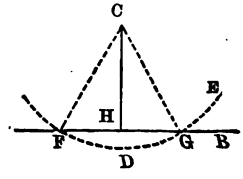
PROP. XII.—PROBLEM.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

(References — Prop. I. 8, 10; post. 3; def. 10, 15.)

Let AB be the given straight line, which may be produced to any length both ways, and C the given point without it.

It is required to draw from C a straight line perpendicular to AB.



CONSTRUCTION

Take any point D, upon the other side of AB,
and from the centre C, at the distance CD, describe the circle EGF, meeting AB in F, G; (post. 3)
bisect FG in H, (I. 10) and join CF, CH, CG.

Then the straight line CH, drawn from the point C, is perpendicular to the given straight line AB.

DEMONSTRATION

Because FH is equal to HG, (constr.)
and HC common to the two triangles FHC, GHC,
the two sides FH, HC, are equal to the two GH, HC, each to each;
and the base CF is equal to the base CG; (def. 15)
therefore the angle CHF *is equal to the angle* CHG; (I. 8)
and they are adjacent angles.
But 'when a straight line standing on a straight line makes the adjacent angles equal to one another, each of them is a right angle; and the straight line which stands upon the other is called a perpendicular to it.' (def. 10.)

Therefore CH is perpendicular to AB,
and it has been drawn from the point C. Q. E. F.

PROP. XIII.—THEOREM.

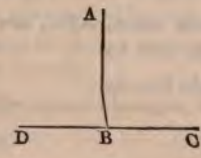
The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

(References—Prop. I. 11; ax. 1, 2; def. 10.)

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

Then the angles CBA, ABD are either two right angles, or are together equal to two right angles.

First, let them be equal to one another.

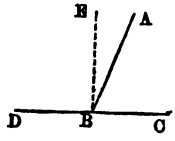


DEMONSTRATION

Then, since the straight line AB, standing on the straight line CD, makes the adjacent angles ABC, ABD equal to one another,
therefore each of them is a right angle; (def. 10)

wherefore the angles ABC, ABD, are two right angles.

Secondly, let them not be equal to one another.



CONSTRUCTION

From the point B draw BE at right angles to CD. (I. 11.)

DEMONSTRATION

Because BE is at right angles to CD, (constr.)
therefore the angles CBE, EBD *are two right angles*; (def. 10)
and because the angle CBE is equal to the two angles CBA, ABE, add to each of these equals the angle EBD;
then the angles CBE, EBD, *are equal to the three angles* CBA, ABE, EBD. (ax. 2.)

Again, because the angle DBA is equal to the two angles DBE, EBA, add to each of these equals the angle ABC;
then the angles DBA, ABC, *are equal to the angles* DBE, EBA, ABC. (ax. 2.)

But the angles CBE, EBD, have been proved to be equal to the same three angles;
and things which are equal to the same thing are equal to one another,
therefore the angles CBE, EBD, *are equal to the angles* DBA, ABC; (ax. 1)
but the angles CBE, EBD are two right angles; (constr.)
therefore the angles DBA, ABC, are together equal to two right angles. (ax. 1.)

Wherefore when a straight line, &c. Q. E. D.

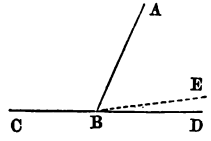
PROP. XIV.—THEOREM.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines shall be in one and the same straight line.

(References—Prop. I. 13; ax. 1, 3.)

At the point B in the straight line AB, let BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD, equal to two right angles.

Then CB shall be in the same straight line with BD.



CONSTRUCTION

For if BD be not in the same right line with BC,
let BE be in the same right line with it.

DEMONSTRATION

Because CBE is assumed to be a straight line, and AB meets it in A,

therefore the adjacent angles ABC, ABE, *must be together equal to two right angles*; (I. 13)

but the angles ABC, ABD, are together equal to two right angles; (hyp.)

therefore the angles CBA, ABE, *must be equal to the angles* CBA, ABD; (ax. 1)

take away the common angle ABC,

therefore the remaining angle ABE *must be equal to the remaining angle* ABD; (ax. 3)

the less to the greater, which is impossible;

therefore BE *is not in the same straight line with* BC.

And, in like manner, it may be proved, that no other can be in the same straight line with it but BD,

therefore BD is in the same straight line with BC.

Wherefore, if at a point, &c. Q. E. D.

PROP. XV.—THEOREM.

If two straight lines cut each other, then the vertical or opposite angles shall be equal.

(References—Prop. I. 13; ax. 1, 3.)

Let the two straight lines AB, CD, cut each other in the point E.

Then the angle AEC shall be equal to the angle DEB, and the angle CEB to the angle AED.

DEMONSTRATION

Because the straight line AE makes with CD the angles CEA, AED,

these angles are together equal to two right angles. (I. 13.)

Again, because the straight line DE makes with AB the angles AED, DEB,

these angles are also together equal to two right angles;

and CEA, AED, have been proved to be together equal to two right angles;

wherefore the angles CEA, AED, *are equal to the angles* AED, DEB; (ax. 1)

take away the common angle AED,

then the remaining angle CEA is equal to the remaining angle DEB. (ax. 3.)

In the same manner it can be shown,

that the angle CEB is equal to the angle AED.

Therefore, if two straight lines, &c. Q. E. D.

Cor. 1. From this it is manifest, that, if two straight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

Cor. 2. And, consequently, that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

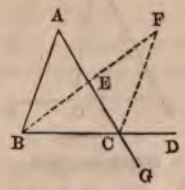
PROP. XVI.—THEOREM.

If one side of a triangle be produced,
then the exterior angle is greater than either of the interior opposite angles.

(References—Prop. I. 3, 4, 10, 15; ax. 9.)

Let ABC be a triangle, and let the side BC be produced to D.

Then the exterior angle ACD shall be greater than either of the angles ABC or CAB.



CONSTRUCTION

Bisect AC in E, (I. 10) and join BE,
produce BE to F, and make EF equal to BE; (I. 3)
and join FC.

DEMONSTRATION

Because AE is equal to EC, and BE to EF;
the two sides AE, EB, are equal to the two CE, EF, each to each;
and the angle AEB is equal to the angle CEF,
because they are opposite, vertical angles; (I. 15)
therefore the base AB *is equal to the base* CF; (I. 4)
and the triangle AEB *to the triangle* CEF,
and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite;
wherefore the angle BAE *is equal to the angle* ECF;
but the angle ECD is greater than the angle ECF;

therefore the angle ACD is greater than the angle BAE.

In the same manner, if the side BC be bisected, and AC be produced to G, it may be demonstrated that the angle BCG,

that is, the angle ACD, (I. 15) **is greater than the angle ABC.**

Therefore, if one side, &c. Q. E. D.

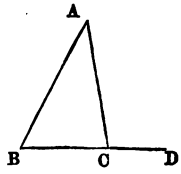
PROP. XVII. — THEOREM.

Any two angles of a triangle are together less than two right angles.

(*References* — Prop. I. 13, 16; ax. 4.)

Let ABC be any triangle.

Then any two of its angles shall be together greater than two right angles.



CONSTRUCTION

Produce the side BC to D.

DEMONSTRATION

Because ACD is the exterior angle of the triangle ABC,

therefore the angle ACD *is greater than the interior and opposite angle* ABC; (I. 16)

to each of these equals add the angle ACB;

then the angles ACD, ACB, *are greater than the angles* ABC, ACB;

but the angles ACD, ACB, are together equal to two right angles; (I. 13)

therefore the angles ABC, BCA, are less than two right angles.

In like manner, it may be proved,

that the angles BAC, ACB, are less than two right angles,

and likewise the angles CAB, ABC.

Therefore any two angles, &c. Q. E. D.

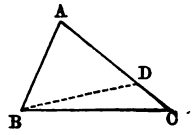
PROP. XVIII. — THEOREM.

The greater side of every triangle is opposite to the greater angle; i.e., in any triangle, if one side be greater than another, then the angle which is opposite the greater side is greater than the angle which is opposite the less.

(References — Prop. I. 3, 5, 16.)

Let ABC be any triangle of which the side AC is greater than the *side AB.*

Then the angle ABC shall be greater than the angle BCA.



CONSTRUCTION

Because the side AC is greater than the side AB,
make AD equal to AB, (I. 3.) and join BD.

DEMONSTRATION

Because the angle ADB is the exterior angle of the triangle BDC,

therefore the angle ADB *is greater than the interior and opposite angle* DCB; (I. 16)

but the angle ADB is equal to the angle ABD, (I. 5)
because the side AB is equal to the side AD,

therefore the angle ABD *is likewise greater than the angle* ACB;

wherefore much more is the angle ABC greater than the angle ACB.

Therefore the greater side, &c. Q. E. D.

PROP. XIX.—THEOREM.

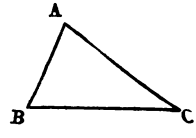
The greater angle of every triangle is subtended by the greater side, i.e., in any triangle, if one angle be greater than another,

then the side which is opposite the greater angle is greater than the side which is opposite to the less.

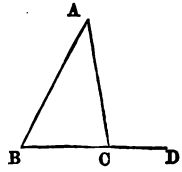
(References—Prop. I. 5, 18.)

Let ABC be any triangle, of which the angle ABC is greater than the angle BCA.

Then the side AC shall be greater than the side AB.



Then any two of its angles shall be together greater than two right angles.



CONSTRUCTION

Produce the side BC to D.

DEMONSTRATION

Because ACD is the exterior angle of the triangle ABC,

therefore the angle ACD *is greater than the interior and opposite angle* ABC; (I. 16)

to each of these equals add the angle ACB;

then the angles ACD, ACB, *are greater than the angles* ABC, ACB;

but the angles ACD, ACB, are together equal to two right angles; (I. 13)

therefore the angles ABC, BCA, are less than two right angles.

In like manner, it may be proved,

that the angles BAC, ACB, are less than two right angles,

and likewise the angles CAB, ABC.

Therefore any two angles, &c. Q. E. D.

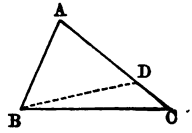
PROP. XVIII. — THEOREM.

The greater side of every triangle is opposite to the greater angle; i.e., in any triangle, if one side be greater than another, then the angle which is opposite the greater side is greater than the angle which is opposite the less.

(References — Prop. I. 3, 5, 16.)

Let ABC be any triangle of which the side AC is greater than the *side AB.*

the angle ABC shall be greater than the angle



CONSTRUCTION

the side AC is greater than the side AB,
D equal to AB, (I. 3.) and join BD.

DEMONSTRATION

the angle ADB is the exterior angle of the triangle BDC,
ɔre the angle ADB *is greater than the interior and opposite angle* B; (I. 16)
ıngle ADB is equal to the angle ABD, (I. 5)
the side AB is equal to the side AD,
ɔre the angle ABD *is likewise greater than the angle* ACB;
efore much more is the angle ABC greater than the ɡle ACB.
:e the greater side, &c. Q. E. D.

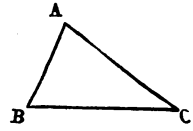
PROP. XIX.—THEOREM.

reater angle of every triangle is subtended by the greater side, i.e., iangle, if one angle be greater than another,
ie side which is opposite the greater angle is greater than the side opposite to the less.

(References—Prop. I. 5, 18.)

BC be any triangle, of which the angle ABC is greater than
ə BCA.

the side AC shall be greater than the side AB.



For, if AC be not greater than AB,
AC must either be equal to AB, or less than it.
But AC is not equal to AB,
because then the angle ABC would be equal to the angle ACB; (I. 5)
but it is not, (hyp.)
therefore AC *is not equal to* AB.

Neither is AC less than AB;
because then the angle ABC would be less than the angle ACB; (I. 18)
but it is not; (hyp.)
therefore AC *is not less than* AB;
and it has been shown that it is not equal to AB;
therefore AC is greater than AB.
Wherefore, in any triangle, &c. Q. E. D.

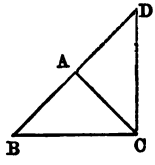
PROP. XX.— THEOREM.

Any two sides of a triangle are together greater than the third side.

(References—Prop. I. 3, 5, 19; ax. 9.)

Let ABC be a triangle.

Then any two sides of it together are greater than the third side, viz.,
the sides BA, AC, greater than the side BC;
AB, BC, greater than AC;
and BC, CA, greater than AB.



CONSTRUCTION

Produce BA to the point D,
and make AD equal to AC; (I. 3) and join DC.

DEMONSTRATION

Because DA is equal to AC,
therefore the angle ADC *is equal to the angle* ACD; (I. 5)
but the angle BCD is greater than the angle ACD;

therefore the angle BCD *is greater than the angle* ADC;

and because in the triangle DCB, the angle BCD is greater than the angle DCB,

and that the side which is opposite the greater angle is greater than that which is opposite the less; (I. 19)

therefore the side DB *is greater than the side* BC;

but DB is equal to BA and AC; (constr.)

therefore the sides BA, AC, are greater than BC.

In the same manner, it may be proved,

that the sides AB, BC, are greater than CA;

and BC, CA, are greater than AB.

Therefore, any two sides, &c. Q. E. D.

PROP. XXI.—THEOREM.

If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle;

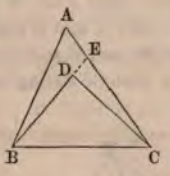
then these shall be less than the other two sides of the triangle, but shall contain a greater angle.

(References — I. 16, 20; ax. 4.)

Let the two straight lines BD, CD, be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it.

Then BD and DC shall be less than the two sides BA, AC, of the triangle,

but shall contain an angle BDC greater than the angle BAC.



CONSTRUCTION

Produce BD to meet AC in E.

DEMONSTRATION

Because two sides of a triangle are greater than the third side, (I. 20)

therefore the two sides BA, AE, *of the triangle* ABE, *are greater than* BE;

to each of these add EC;
then the sides BA, AC, *are greater than* BE, EC. (ax. 4.)

Again, because the two sides CE, ED, of the triangle CED, are greater than CD,
add DB to each of these;
then the sides CE, EB, *are greater than* CD, DB; (ax. 4)
but it has been shown that BA, AC, are greater than BE, EC,
much more then are BA, AC, greater than BD, DC.

Again, because the exterior angle of a triangle is greater than the interior and opposite angle; (I. 16)
therefore the exterior angle BDC *of the triangle* CDE *is greater than* CED;
for the same reason,
the exterior angle CEB *of the triangle* ABE *is greater than* BAC;
and it has been shown that the angle BDC is greater than CEB;
much more then is the angle BDC greater than the angle BAC.

Therefore, if from the ends, &c. Q. E. D.

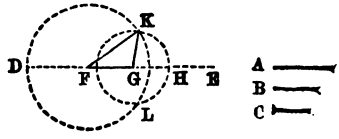
PROP. XXII.—PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but of which any two whatever must be greater than the third.

(References—Prop. I. 3; post. 3; ax. 1; def. 15.)

Let A, B, C be three given right lines, of which any two whatever are greater than the third — viz., A and B together greater than C; A and C together greater than B; and B and C greater than A.

It is required to make a triangle having its sides equal to A, B, C, each to each.



CONSTRUCTION

Take a straight line DE terminated at the point D, but unlimited ***towards E,***

make DF equal to A, FG equal to B, and GH equal to C; (I. 3)
and from the centre F, at the distance FD, describe the circle DKL; (post. 3)
and from the centre G, at the distance GH, describe the circle HLK;
and join KF, KG.

Then the triangle KFG shall have its sides equal to the three straight lines A, B, C.

DEMONSTRATION.

Because the point F is the centre of the circle DKL,
therefore FD *is equal to* FK; (def. 15)
but FD is equal to the straight line A; (constr.)
therefore FK *is equal to* A. (ax. 1)

Again, because G is the centre of the circle LKH,
therefore GH *is equal to* GK; (def. 15)
but GH is equal to C; (constr.)
therefore GK *is equal to* C;
and FG is equal to B; (constr.)
therefore the three straight lines KF, FG, GK, *are equal to the three* A, B, C.

And therefore the triangle KFG has its sides KF, FG, GK, equal to the given straight lines A, B, C. Q. E. F.

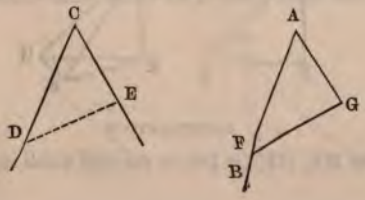
PROP. XXIII.—Problem.

At a given point in a given straight line to construct a rectilineal angle equal to a given rectilineal angle.

(References—Prop. I. 8, 22.)

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle.

It is required to make an angle at the point A in the straight line AB, equal to the rectilineal angle DCE.



CONSTRUCTION

Take in CD, CE, any points D, E, and join DE;
make the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, CE, so that CD be equal to AF, CE to AG, and DE to FG. (I. 22.)

Then the angle FAG shall be equal to the angle DCE.

DEMONSTRATION

Because DC, CE, are equal to FA, AG, each to each,
and the base DE to the base FG; (constr.)

therefore the angle DCE is equal to the angle FAG. (I. 8.)

Wherefore, at the given point A in the straight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Q. E. F.

PROP. XXIV.—THEOREM.

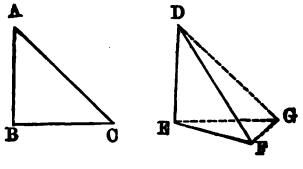
If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other;

then the base of that which has the greater angle, shall be greater than the base of the other.

(References—Prop. I. 3, 4, 5, 19, 23; ax. 9.)

Let ABC, DEF, be two triangles which have the two sides AB, AC, equal to the two DE, DF, each to each—viz., AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF.

Then the base BC shall be greater than the base EF.



CONSTRUCTION

Of the two sides DE, DF, let DE be the side which is not greater than *the other,*

and at the point D, in the straight line DE, make the angle EDG equal to the angle BAC; (I. 23)

and make DG equal to AC or DF, (I. 3) and join EG, GF.

DEMONSTRATION

Because AB is equal to DE, and AC to DG,

the two sides BA, AC, are equal to the two ED, DG, each to each,

and the angle BAC is equal to the angle EDG; (constr.)

therefore the base BC *is equal to the base* EG. (I. 4.)

And, because DG is equal to DF,

therefore the angle DFG *is equal to the angle* DGF; (I. 5)

but the angle DGF is greater than the angle EGF,

therefore the angle DFG is greater than the angle EGF;

much more then is the angle EFG *greater than the angle* EGF.

And, because the angle EFG of the triangle EFG, is greater than its angle EGF,

and that the greater side is opposite to the greater angle; (I. 19)

therefore the side EG *is greater than the side* EF;

but EG has been proved to be equal to BC;

therefore BC is greater than EF.

Therefore, if two triangles, &c. Q. E. D.

PROP. XXV.—THEOREM.

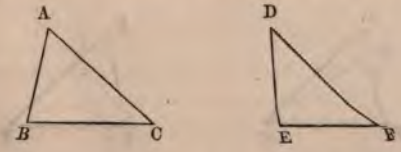
If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other;

then the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

(References—Prop. I. 4, 24.)

Let ABC, DEF, be two triangles which have the two sides AB, AC, equal to the two sides DE, DF, each to each—viz., AB equal to DE, and AC to DF; but the base CB greater than the base EF.

Then the angle BAC shall be greater than the angle EDF.



DEMONSTRATION

For if the angle BAC be not greater than the angle DEF, it must either be equal to it or less.

Now, if the angle BAC be equal to the angle EDF,

then must the base BC *be equal to the base* EF ; (I. 4)

but it is not ; (hyp.)

therefore the angle BAC *is not equal to the angle* EDF.

Again, if the angle BAC be less than the angle EDF,

then must the base BC *be less than the base* EF; (I. 24)

but it is not ; (hyp.)

therefore the angle BAC *is not less than the angle* EDF;

and it has been proved it is not equal to it ;

therefore the angle BAC is greater than the angle EDF.

Wherefore, if two triangles, &c. Q. E. D.

PROP. XXVI.— THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side ; viz., either the side adjacent to the equal angles in each, or the side opposite to them ;

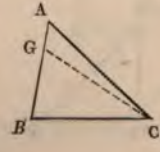
then the other sides shall be equal, each to each, and also the third angle of the one equal to the third angle of the other.

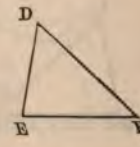
(References — Prop. I. 3, 4, 16 ; ax. 1.)

Let the two triangles ABC, DEF, have the angles ABC, BCA, equal to the angles DEF, EFD, each to each ; also one side equal to one side.

First. Let the sides adjacent to the equal angles in each be equal; viz., BC equal to EF.

Then shall the side AB be equal to the side DE, the side AC to the side DF, and the angle BAC to the angle EDF.





CONSTRUCTION

For if AB be not equal to DE,
one of them is greater than the other.
Let AB be the greater of the two,
and make BG equal to DE, (I. 3) and join GC.

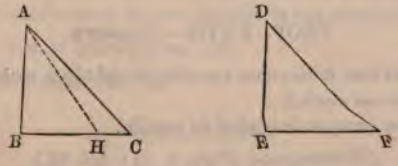
DEMONSTRATION

Then in the two triangles GBC, DEF,
because BG is assumed to be equal to DE,
and BC is equal to EF, (hyp.)
the two sides GB, BC, must be equal to the two DE, EF, each to each;
and the angle GBC is equal to the angle DEF; (hyp.)
therefore the base GC must be equal to the base DF,
and the triangle GBC to the triangle DEF,
and the remaining angles of the one equal to the remaining angles of the other, each to each, to which the equal sides are opposite;
therefore the angle GCB *must be equal to the angle* DFE; (I. 4)
but the angle DFE is, by the hypothesis, equal to the angle BCA;
wherefore also the angle BCG *must be equal to the angle* BCA, (ax. 1)
the less to the greater, which is impossible;
therefore AB *is not unequal to* DE,
that is, AB is equal to DE.

Therefore in the triangles ABC, DEF,
because the two AB, BC, are equal to the two DE, EF, each to each;
and the angle ABC is equal to the angle DEF; (hyp.)
therefore the base AC is equal to the base DF,
and the third angle BAC to the third angle EDF. (I. 4.)

Next, let the sides which are opposite to the equal angles in each triangle be equal to one another, viz., AB to DE.

Then likewise in this case the other sides shall be equal, AC to DF, and BC to EF; and also the third angle BAC to the third angle EDF.



CONSTRUCTION

For if BC be not equal to EF,
one of them must be greater than the other.
Let BC be the greater,
make BH equal to EF, (I. 3) and join AH.

DEMONSTRATION

Then, in the two triangles ABH, DEF,
because BH is assumed to be equal to EF,
and AB is equal to DE, (hyp.)
the two sides AB, BH, must be equal to the two DE, EF, each to each;
and the angle ABH is equal to the angle DEF; (hyp.)
therefore the base AH must be equal to the base DF,
and the triangle ABH to the triangle DEF,
and the remaining angles of the one equal to the remaining angles of the other, each to each, to which the equal sides are opposite;
therefore the angle BHA *must be equal to the angle* BCA, (I. 4)
that is, the exterior angle, BHA, of the triangle ABC, must be equal to its interior and opposite angle BCA,
which is impossible; (I. 16)
wherefore BC *is not unequal to* EF,
that is, BC is equal to EF.

Therefore in the triangles ABC, DEF,
because AB is equal to DE, (hyp.)
and BC has been shown to be equal to EF,
the two AB, BC, are equal to the two DE, EF, each to each;
and the angle ABC is equal to the angle DEF; (hyp.)
therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. (I. 4.)

Wherefore, if two triangles, &c. Q. E. D.

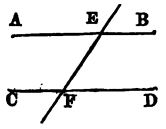
PROP. XXVII.—THEOREM.

If a straight line, falling upon two other straight lines, make the alternate angles equal to one another,
then these two straight lines shall be parallel.

(References—Prop. I. 16; def. 35.)

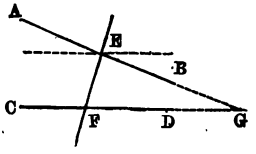
Let the straight line EF, which falls upon the two straight lines, AB, CD, make the alternate angles, AEF, EFD, equal to one another.

Then AB shall be parallel to CD.



CONSTRUCTION

For, if AB be not parallel to CD,
AB and CD being produced, shall meet either towards B, D, or towards A, C;
let them be produced and meet towards B, D in the point G.



DEMONSTRATION

Then GEF must be a triangle,

and its exterior angle AEF *must be greater than the interior and opposite angle* EFG; (I. 16)

but the angle AEF is equal to the angle EFG; (hyp.)

therefore the angle AEF *must be both greater than, and equal to the angle* EFG;

which is impossible.

Therefore AB, CD, *being produced, do not meet towards* B, D.

In like manner it may be demonstrated that
AB, CD, *do not meet towards* A, C.

But those straight lines, in the same plane, which meet neither way, though produced ever so far, are parallel to one another; (def. 35)

therefore AB is parallel to CD.

Wherefore, if a straight line, &c. Q. E. D.

PROP. XXVIII.—THEOREM.

If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite, upon the same side of the line, or make the interior angles upon the same side, together equal to two right angles;

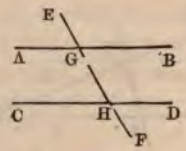
then the two straight lines shall be parallel to one another.

(References — Prop. I. 13, 15, 27; ax. I. 3.)

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side;

or make the interior angles on the same side BGH, GHD, together equal to two right angles.

Then AB shall be parallel to CD.



DEMONSTRATION

Because the angle EGB is equal to the angle GHD, (hyp.)
and the angle EGB is equal to the angle AGH, (I. 15)
therefore the angle AGH *is equal to the angle* GHD; (ax. 1)
and they are alternate angles,
therefore AB is parallel to CD. (I. 27)

Again, because the angles BGH, GHD, are equal to two right angles; (hyp.)
and that the angles AGH, BGH, are also equal to two right angles; (I. 13)
therefore the angles AGH, BGH, *are equal to the angles* BGH, GHD; (ax. 1)
take away the common angle BGH;
therefore the remaining angle AGH *is equal to the remaining angle* GHD; (ax. 3)
and they are alternate angles;
therefore AB is parallel to CD. (I. 27)

Wherefore, if a straight line, &c. Q.E.D.

PROP. XXIX.— THEOREM.

If a straight line fall upon two parallel straight lines, then it makes the alternate angles equal to each other; and the exterior angle equal to the interior and opposite angle upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

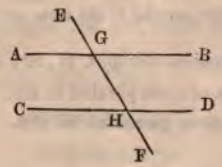
(References — Prop. I. 13, 15; ax. I. 2, 4, 12.)

Let the right line EF, fall on the parallel lines AB, CD.

Then the alternate angle AGH shall be equal to the alternate angle GHD;

and the exterior angle EGB shall be equal to the interior and opposite angle GHD, on the same side of the line EF;

and the two interior angles BGH, GHD, on the same side, shall be together equal to two right angles.



DEMONSTRATION

For if the angle AGH be not equal to GHD,
one of them must be greater than the other.
Let AGH be the greater.
Then because the angle AGH is assumed to be greater than the angle GHD,
add to each of them the angle BGH;
therefore the angles AGH, BGH, *must be greater than the angles* BGH, GHD; (ax. 4)
but the angles AGH, BGH, are equal to two right angles; (I. 13)
therefore the angles BGH, GHD, *must be less than two right angles;*
but 'those straight lines which with another straight line falling upon them, make the interior angles on the same side less than two right angles, will meet together if continually produced;' (ax. 12)
therefore the straight lines AB, CD, *must meet if produced far enough.*
but they never meet, since they are parallel by the hypothesis;
therefore the angle AGH *is not unequal to the angle* GHD,
that is, the angle AGH is equal to the angle GHD, which are the alternate angles.

But the angle AGH is equal to the angle EGB; (I. 15)

therefore the angle EGB is equal to the angle GHD, (ax. 1)

that is, the exterior angle is equal to the interior and opposite.

Add to each of these the angles BGH;

therefore the angles EGB, BGH, *are equal to the angles* BGH, GHD; (ax. 2)

but EGB, BGH are equal to two right angles; (I. 13)

therefore also BGH, GHD, are equal to two right angles, (ax. 1)

which are the two interior angles.

Wherefore, if a straight line, &c. Q. E. D.

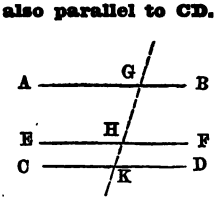
PROP. XXX.—THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

(References—Prop. I. 27, 29; ax. 1.)

Let AB, CD, be each of them parallel to EF.

Then AB shall be also parallel to CD.



CONSTRUCTION

Let the straight line GHK cut AB, EF, CD.

DEMONSTRATION

Because GHK cuts the parallel straight lines AB, EF,

therefore the angle AGH *is equal to the angle* GHF. (I. 29.)

Again, because the straight line GHK cuts the parallel straight lines EF, CD,

therefore the angle GHF *is equal to the angle* GKD;

and it was shown that the angle AGK is equal to the angle GHF;

therefore also the angle AGK *is equal to the angle* GKD; (ax. 1)

and they are alternate angles;

***therefore* AB is parallel to CD.** (I. 27.)

Wherefore, straight lines, &c. Q. E. D.

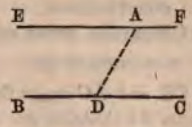
PROP. XXXI.—Problem.

To draw a straight line through a given point, parallel to a given straight line.

(References—Prop. I. 23, 27.)

Let A be the given point, and BC the given right line.

It is required to draw a straight line through the point A, parallel to BC.



CONSTRUCTION

In BC take any point D, and join AD;

at the point A, in the straight line AD, make the angle DAE equal to the angle ADC; (I. 23)

and produce the straight line EA to F.

Then EF shall be parallel to BC.

DEMONSTRATION

Because the straight line AD, which meets the two straight lines BC, EF,

makes the alternate angles EAD, ADC, equal to one another,

therefore EF is parallel to BC. (I. 27.)

Wherefore the straight line EAF is drawn through the given point A parallel to the given straight line BC. Q. E. F.

PROP. XXXII.—Theorem.

If the side of any triangle be produced;

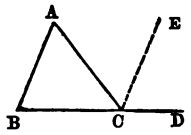
then the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

(References—Prop. I. 13, 29, 31; ax. 2.)

Let ABC be a triangle, and let one of its sides BC be produced to D.

Then the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC;

and the three interior angles of the triangle — viz., ABC, BCA, CAB, are together equal to two right angles.



CONSTRUCTION

Through the point C draw CE parallel to the straight line AB. (I. 31.)

DEMONSTRATION

Because AB is parallel to CE, and AC meets them,

therefore the alternate angles BAC, ACE, *are equal.* (I. 29.)

Again, because AB is parallel to CE, and BD falls upon them,

therefore the exterior angle ECD *is equal to the interior and opposite angle* ABC; (I. 29)

but the angle ACE was shown to be equal to the angle BAC;

therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC. (ax. 2.)

To each of these equals add the angle ACB;

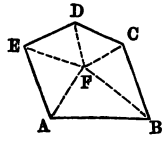
therefore the angles ACD, ACB, *are equal to the three angles* CBA, BAC, ACB;

but the angles ACD, ACB, are equal to two right angles; (I. 13)

therefore also the angles CBA, BAC, ACB, are equal to two right angles. (ax. 1.)

Wherefore, if a side of a triangle, &c. Q. E. D.

Cor. 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



DEMONSTRATION

For, any rectilineal figure ABCDE can, by drawing straight lines from *a point F within* the figure to each angle, be divided into as many *triangles as the* figure has sides.

By the proposition, the angles of each triangle are equal to two right angles;

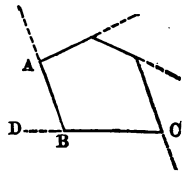
therefore all the angles of the triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure;

but the same angles are equal to the angles of the figure, together with the angles at the point F;

and the angles at F, which is the common vertex of all the triangles, are equal to four right angles, (I. 15, cor. 2)

therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Cor. 2. *All the exterior angles of any rectilineal figure are together equal to four right angles.*



DEMONSTRATION

For, because each interior angle ABC, together with its adjacent exterior angle ABD, are equal to two right angles, (I. 13)

therefore all the interior angles, together with all the exterior angles, are equal to twice as many right angles as there are angles or sides.

but all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides; (cor. 1)

therefore all the interior angles, together with all the exterior angles, are equal to all the interior angles and four right angles; (ax. 1)

take away the interior angles which are common;

therefore all the exterior angles are equal to four right angles. (ax. 3.)

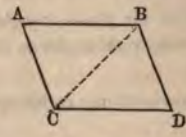
PROP. XXXIII.—THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

(References—Prop. I. 4, 27, 29.)

Let AB, CD, be two equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD.

Then AC, BD, shall be equal and parallel.



CONSTRUCTION

Join the opposite points B and C.

DEMONSTRATION

Because AB is parallel to CD, and BC meets them,
therefore the alternate angles ABC, BCD, *are equal*; (I. 29)
and because AB is equal to CD, (hyp.) and BC common to the two triangles ABC, DCB,
the two sides AB, BC, are equal to the two DC, CB;
and the angle ABC has been shown to be equal to the angle BCD;
therefore the base AC is equal to the base BD, (I. 4)
and the triangle ABC to the triangle BCD,
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle ACB *is equal to the angle* CBD.
And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD, equal to one another,
therefore AC is parallel to BD, (I. 27)
and AC was shown to be equal to BD.

Therefore, straight lines, &c. Q. E. D.

PROP. XXXIV.—THEOREM.

The opposite sides and angles of parallelograms are equal to each other, and the diameter bisects them, that is, divides them into two equal parts.

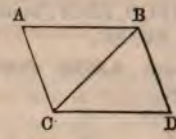
N.B. A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.

(References—Prop. I. 4, 26, 29; ax. 2.)

Let ACDB be a parallelogram, of which BC is a diameter.

Then the opposite sides and angles of the figure shall be equal to one another;

and the diameter shall bisect it.



DEMONSTRATION

Because AB is parallel to CD, and BC meets them,

therefore the alternate angles ABC, BCD, *are equal to one another;* (I. 29)

and because AC is parallel to BD, and BC meets them,

therefore the alternate angles ACB, CBD, *are equal to one another.*

Wherefore in the two triangles ABC, CBD,

because the two angles ABC, BCA, in the one, are equal to the two angles BCD, CBD, in the other, each to each,

and one side BC, which is adjacent to their equal angles, is common to the two triangles;

therefore their other sides shall be equal, each to each,

and the third angle of the one to the third angle of the other (I. 26)—viz.,

the side AB *to the side* CD, *and* AC *to* BD,

and the angle BAC *to the angle* BDC.

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB;

therefore the whole angle ABD *is equal to the whole angle* ACD; (ax. 2)

and the angle BAC has been shown to be equal to the angle BDC;

therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diameter shall bisect it.

For because AB is equal to CD, and BC common,

the two AB, BC, are equal to the two DC, CB, each to each;

and the angle ABC has been proved equal to the angle BCD;

therefore the triangle ABC *is equal to the triangle* BCD; (I. 4)

and the diameter BC divides the parallelogram ACDB *into two* equal parts. Q.E.D.

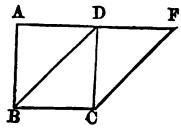
PROP. XXXV.—THEOREM.

Parallelograms upon the same base, and between the same parallels, are equal to each other.

(References—Prop. I. 4, 29, 34; ax. 1, 3, 6.)

Let the parallelograms ABCD, EBCF, be on the same base BC, and between the same parallels AF, BC.

Then the parallelogram ABCD shall be equal to the parallelogram EBCF.

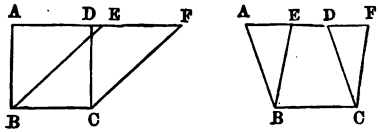


DEMONSTRATION

If the sides AD, DF, of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D,

it is plain that each of the parallelograms ABCD, DBCF, *is double of the triangle* BDC; (I. 34)

and therefore the parallelograms ABCD, DBCF, are equal to one another. (ax. 6.)



But if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point;

then, because ABCD is a parallelogram,

therefore AD *is equal to* BC; (I. 34)

for the same reason EF is equal to BC;

wherefore AD *is equal to* EF; (ax. 1)

and DE is common;

*therefore the whole, or remainder** AE, *is equal to the whole, or remainder* DF; (ax. 2 or 3)

---

* The words 'whole or remainder' are used to suit both figures (2 and 3), since, in figure 2, DE is *added to* AD and EF, which therefore gives the whole AE; and *in figure 3, DE is taken away from* AD and EF, which therefore leaves the remainder *AE.*

and AB is equal to DC. (I. 34.)
Wherefore, in the triangles EAB, FDC,
because the two EA, AB, are equal to the two FD, DC, each to each;
and the exterior angle FDC is equal to the interior EAB, (I. 29)
therefore the base EB is equal to the base FC,
and the triangle EAB *equal to the triangle* FDC. (I. 4.)

Take the triangle FDC from the trapezium ABCF,
and from the same trapezium take the triangle EAB;
the remainders are therefore equal, (ax. 3)
that is, the parallelogram ABCD is equal to the parallelogram EBCF.

Therefore, parallelograms upon the same base, &c. Q. E. D.

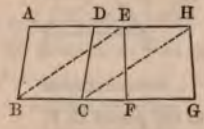
PROP. XXXVI.—THEOREM.

Parallelograms upon equal bases and between the same parallels are equal to one another.

(References—Prop. I. 33, 34, 35; ax. 1.)

Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG.

Then the parallelogram ABCD shall be equal to the parallelogram EFGH.



CONSTRUCTION

From B to E, draw the straight line BE, and from C to H, the straight line CH.

DEMONSTRATION

Because BC is equal to FG, (hyp.) and FG to EH, (I. 34)
therefore BC *is equal to* EH; (ax. 1)
and they are parallels, and joined towards the same parts by the straight lines BE, CH; (hyp.)
but 'straight lines which join equal and parallel straight lines towards the same parts, are themselves equal and parallel;' (I. 33)
therefore EB, CH, are both equal and parallel;
wherefore EBCH is a parallelogram. (I. 34, def.)

Then because the parallelograms ABCD, EBCH, are upon the same base BC,

and between the same parallels BC, AH;

therefore the parallelogram ABCD *is equal to the parallelogram* EBCH; (I. 35)

for the like reason, the parallelogram EFGH is equal to the same EBCH;

therefore the parallelogram ABCD is equal to the parallelogram EFGH. (ax. 1.)

Wherefore, parallelograms, &c. Q. E. D.

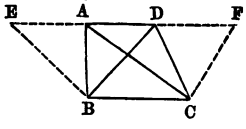
PROP. XXXVII.—THEOREM.

Triangles upon the same base, and between the same parallels, are equal to one another.

(References—Prop. I. 31, 34, 35; ax. 7.)

Let the triangles ABC, DBC, be on the same base BC, and between the same parallels AD, BC.

Then the triangle ABC shall be equal to the triangle DBC.



CONSTRUCTION

Produce AD both ways to the points E, F,

and through B draw BE parallel to CA, and through C draw CF parallel to BD. (I. 31.)

DEMONSTRATION

Then, each of the figures EBCA, DBCF, is a parallelogram; (I. 34, def.)

and because the parallelograms EBCA, DBCF, are upon the same base BC,

and between the same parallels BC, EF,

therefore the parallelogram EBCA *is equal to the parallelogram* DBCF; (I. 35)

and because the diameter AB bisects the parallelogram EBCA;

therefore the triangle ABC *is the half of* EBCA; (I. 34)

and because the diameter DC bisects the parallelogram DBCF,
therefore the triangle DBC *is the half of* DBCF;
but 'things which are halves of the same are equal to one another,' (ax. 7)
therefore the triangle ABC is equal to the triangle DBC.
Wherefore, triangles, &c. Q. E. D.

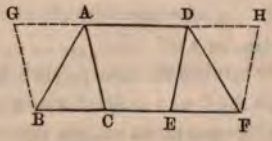
PROP. XXXVIII.— THEOREM.

Triangles upon equal bases and between the same parallels are equal to one another.

(References — Prop. I. 31, 34, 36; ax. 7.)

Let the triangles ABC, DEF, be on the equal bases BC, EF, and between the same parallels AD, BF.

Then the triangle ABC shall be equal to the triangle DEF.



CONSTRUCTION

Produce AD both ways to the points G, H,
and through B draw BG parallel to CA, and through F draw FH parallel to ED. (I. 31.)

DEMONSTRATION

Then, each of the figures GBCA, DEFH, is a parallelogram; (I. 34 def.)
and because they are upon equal bases BC, EF,
and between the same parallels BF, GH,
therefore these parallelograms are equal to one another. (I. 36.)
And because the diameter AB bisects the parallelograms GBCA,
therefore the triangle ABC *is the half of* GBCA; (I. 34)
and because the diameter DF bisects the parallelogram DEFH,
therefore the triangle DEF *is the half of* DEFH;
but 'things which are halves of the same are equal to one another,' (ax 7.)
therefore the triangle ABC is equal to the triangle DEF.
Wherefore, triangles, &c. Q. E. D.

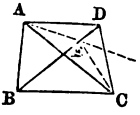
PROP. XXXIX.— THEOREM.

Equal triangles upon the same base and on the same side of it, are between the same parallels.

(References — Prop. I. 31, 37; ax. 1.)

Let the equal triangles ABC, DBC, be upon the same base BC, and on the same side of it.

Then they shall be between the same parallels.



CONSTRUCTION

From A to D, draw the straight line AD.

Then AD is parallel to BC,

For, if it be not, through the point A draw AE parallel to BC, (I. 31) and join EC.

DEMONSTRATION

Then, because the triangles ABC, EBC, are on the same base BC, and are assumed to be between the same parallels BC, AE;

therefore the triangle ABC *must be equal to the triangle* EBC ; (I. 37)

but the triangle ABC is equal to the triangle DBC ; (hyp.)

therefore the triangle DBC *must be equal to the triangle* EBC, (ax. 1)

the greater to the less, which is impossible;

therefore AE *is not parallel to* BC.

In the same manner, it can be demonstrated that no other line but AD is parallel to BC;

therefore AD is parallel to BC.

Wherefore, equal triangles upon, &c. Q.E.D.

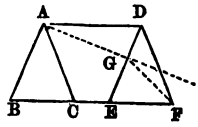
PROP. XL.— THEOREM.

Equal triangles upon equal bases in the same straight line and towards the same parts are between the same parallel.

(References — Prop. I. 31, 38; ax. 1.)

Let the equal triangles ABC, DEF, be upon the equal bases BC, EF, in the same straight line BF, and towards the same parts.

Then the triangles ABC, DEF, shall be between the same parallels.



CONSTRUCTION

From A to D, draw the straight line AD;

then AD is parallel to BF.

For, if it be not, through A draw AG parallel to BF, (I. 31) and join GF.

DEMONSTRATION

Because the triangles ABC, GEF, are upon equal bases BC, EF, and are assumed to be between the same parallels BF, AG;

therefore the triangle ABC *must be equal to the triangle* GEF; (I. 38)

but the triangle ABC is equal to the triangle DEF, (hyp.)

therefore also the triangle DEF *must be equal to the triangle* GEF, (ax. 1)

the greater to the less, which is impossible;

therefore AG *is not parallel to* BF.

And in the same manner it can be demonstrated that there is no other parallel to it but AD;

therefore AD is parallel to BF.

Wherefore equal triangles, &c. Q. E. D.

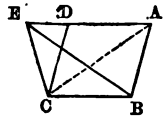
PROP. XLI.— THEOREM.

If a parallelogram and a triangle be upon the same base and between the same parallels; the parallelogram shall be double of the triangle.

(References — Prop. I. 34, 37; ax. 1.)

Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE.

Then the parallelogram ABCD shall be double of the triangle EBC.



CONSTRUCTION

From A to C draw the straight line AC

DEMONSTRATION

Because the triangles ABC, ECB are upon the same base BC, and between the same parallels BC, AE,

therefore the triangle ABC *is equal to the triangle* EBC; (I. 37)

but the diameter AC bisects the parallelogram ABCD,

therefore the parallelogram ABCD *is double of the triangle* ABC, (I. 34)

wherefore also ABCD is double of the triangle EBC. (AX. 1)

Therefore if a parallelogram, &c. Q. E. D.

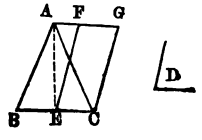
PROP. XLII.— PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

(References — Prop. I. 10, 23, 31, 38, 41; AX. 6; def. I. 34.)

Let ABC be the given triangle, and D the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



CONSTRUCTION

Bisect BC in E, (I. 10) join AE,

and at the point E in the straight line EC, make the angle CEF equal to D; (I. 23)

through A draw AFG parallel to EC, and through C draw CG parallel *to EF.* (I. 31.)

Then FECG is the parallelogram required.

DEMONSTRATION

Because the base BE is equal to the base EC, and BC parallel to AG, therefore the triangle ABE is equal to the triangle AEC; (I. 38)

and therefore the triangle ABC *is double of the triangle* AEC;

but because the parallelogram FECG and the triangle AEC are on the same base EC, and between the same parallels EC, AG,

therefore the parallelogram FECG *is double of the triangle* AEC; (I. 41)

but 'things which are double of the same are equal to one another' (ax. 6)

therefore the parallelogram FECG is equal to the triangle ABC,

and it has one of its angles CEF equal to the given angle D. (constr.)

Wherefore there has been described a parallelogram FECG equal to a given triangle ABC, having one of its angles CEF equal to the given angle D. Q. E. F.

PROP. XLIII.— THEOREM.

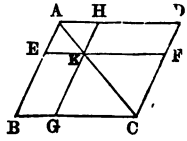
The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

(References — Prop. I. 34; ax. 2, 3.)

Let ABCD be a parallelogram, of which the diameter is AC; and EH, FG, the parallelograms about AC, *that is, through which* AC *passes*;

and let BK, KD be the other parallelograms, which make up the whole figure ABCD, and are therefore called the complements.

Then the complement BK shall be equal to the complement KD.



DEMONSTRATION

Because ABCD is a parallelogram, and AC its diameter,

therefore the triangle ABC *is equal to the triangle* ADC; (I. 34)

and because EKHA is a parallelogram, and AK its diameter,

therefore the triangle AEK *is equal to the triangle* AHK;

and for the same reason,

the triangle KGC *is equal to the triangle* KFC.

And, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC;

therefore the two triangles AEK, KGC, *are equal to the two triangles* AHK, KFC; (ax. 2)

but the whole triangle ABC is equal to the whole triangle ADC;

therefore the remaining complement BK is equal to the remaining complement KD. (ax. 3.)

Wherefore, the complements, &c. Q. E. D.

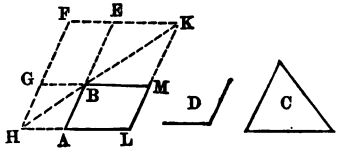
PROP. XLIV.— PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

(References — Prop. I. 15, 29, 31, 42, 43; ax. 9. 12.)

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle.

It is required to apply to the straight line AB, a parallelogram equal to the triangle C, and having an angle equal to D.



CONSTRUCTION

Make the parallelogram BEFG equal to the triangle C, (I. 42)
and having the angle EBG equal to the angle D,
so that BE be in the same straight line with AB;
produce FG to H,
and through A draw AH parallel to BG or EF, (I. 31) and join HB.

Then because the straight line HF falls upon the parallels AH, EF,
therefore the angles AHF, HFE, are together equal to the two *right angles*; (I. 29)

wherefore the angles BHF, HFE, *are less than two right angles;* (ax. 9)
but 'straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet if produced far enough;' (ax. 12)
therefore HB, FE, *shall meet, if produced;*
let them meet in K,
and through K draw KL parallel to EA or FH, (I. 31)
and produce HA, GB, to the points L, M.

Then LB shall be the parallelogram required.

DEMONSTRATION

Because HLKF is a parallelogram, of which the diameter is HK,
and AG, ME, are the parallelograms about HK,
also LB, BF, are the complements;
therefore the complement LB *is equal to the complement* BF; (I. 43)
but the complement BF is equal to the triangle C; (constr.)
wherefore LB is equal to the triangle C.
And because the angle GBE is equal to the angle ABM, (I. 15)
and likewise to the angle D,
therefore the angle ABM is equal to the angle D.
Wherefore the parallelogram LB is applied to the straight line AB, and is equal to the triangle C, and has the angle ABM equal to the angle D. Q. E. F.

PROP. XLV.—Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

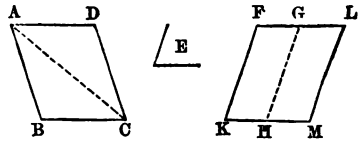
(References—Prop. I. 14, 29, 30, 42, 44; ax. 1, 2; I. 34, def.)

Let ABCD be the given rectilineal figure, and E the given rectilineal angle.

It is required to describe a parallelogram equal to ABCD, and having an angle eqnal to E.

CONSTRUCTION

Join AC, and describe the parallelogram FH equal to the triangle ABC, and having the angle HKF equal to the angle E; (I. 42)
and to the straight line GH apply the parallelogram GM equal to the triangle ADC, having the angle GHM equal to the angle E. (I. 44.)
Then the figure FKML shall be the parallelogram required.



DEMONSTRATION

Because the angle E is equal to each of the angles FKH, GHM,
therefore the angle FKH *is equal to the angle* GHM;
add to each of these the angle KHG;
therefore the angles FKG, KHG, *are equal to the angles* KHG, GHM;
but FKH, KHG are equal to two right angles; (I. 29)
therefore also KHG, GHM, *are equal to two right angles;*
and because at the point H in the straight line GH, the two straight lines KH, HM, upon the opposite sides of it, make the adjacent angles equal to two right angles,
therefore KH *is in the same straight line with* HM. (I. 14.)

And because the straight line HG meets the parallels KM, FG,
therefore the alternate angles MHG, HGF, *are equal;* (I. 29)
add to each of these the angle HGL;
therefore the angles MHG, HGL, *are equal to the angles* HGF, HGL;
but the angles MHG, HGL, are equal to two right angles; (I. 29)
wherefore also the angles HGF, HGL, *are equal to two right angles,*
and therefore FG *is in the same straight line with* GL. (I. 14.)

And because KF is parallel to HG, and HG to ML,
therefore KF *is parallel to* ML; (I. 30)
and KM, FL, are parallels;
wherefore KFLM *is a parallelogram;* (I. 34, def.)
and because the triangle ABC is equal to the parallelogram HF, and the triangle ADC to the parallelogram GM;
therefore the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM.

Wherefore the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, having the angle FKM equal to the angle E. Q. E. F.

Cor. From this it is manifest how to apply to a given right line a *parallelogram,* which shall have an angle equal to a given rectilineal

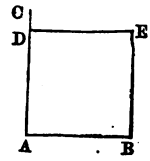
angle, and shall be equal to a given rectilineal figure—viz., by applying to the given straight line a parallelogram equal to the first triangle ABC, and having an angle equal to the given angle.

PROP. XLVI.—Problem.

To describe a square upon a given straight line.

(References—Prop. I. 3, 11, 29, 31, 34; ax. 1, 3; def. 30.)

Let AB be the given straight line.
It is required to describe a square upon AB.



CONSTRUCTION

From the point A draw AC at right angles to AB, (I. 11)
and make AD equal to AB, (I. 3)
through the point D draw DE parallel to AB, (I. 31)
and through B draw BE parallel to AD.

Then ADEB shall be the square required.

DEMONSTRATION

Because DE is parallel to AB, and BE parallel to AD; (constr.)
therefore ADEB is a parallelogram; (I. 34, def.)
wherefore AB *is equal to* DE, *and* AD *to* BE; (I. 34)
but BA is equal to AD; (constr.)
therefore the four straight lines BA, AD, DE, EB, are equal to one another,
and the parallelogram ADEB *is equilateral.*

Likewise all its angles are right angles;
for, since the straight line AD meets the parallels AB, DE,
therefore the angles BAD, ADE *are equal to two right angles;* (I. 29)
but BAD is a right angle; (constr.)
therefore also ADE *is a right angle;* (ax. 3)
but 'the opposite angles of parallelograms are equal;' (I. 34)
therefore each of the opposite angles ABE, BED, is a right angle;

wherefore the parallelogram ADEB *is rectangular.*

And it has been demonstrated that it is equilateral;

therefore ADEB is a square,

and it is described upon the given straight line AB. Q. E. F.

Cor. Hence every parallelogram that has one right angle has all its angles right angles.

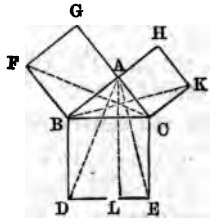
PROP. XLVII.—THEOREM.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

(References—Prop. I. 4, 14, 31, 41, 46; ax. 2, 6, 11; def. 30.)

Let ABC be a right-angled triangle, having the right angle BAC.

Then the square described upon the side BC, shall be equal to the squares described upon BA, AC.



CONSTRUCTION

On BC describe the square BDEC, (I. 46)

and on BA, AC, the squares GB, HC;

through A draw AL parallel to BD, or CE, (I. 31) and join AD, FC.

DEMONSTRATION

Then, because each of the angles BAC, BAG, is a right angle, (hyp., def. 30)

the two straight lines AC, AG, upon the opposite sides of AB, make with it at the point A, the adjacent angles equal to two right angles;

therefore CA *is in the same straight line with* AG; (I. 14)

for the same reason,

AB and AH are in the same straight line.

And because the angle DBC is equal to the angle FBA, each of them being a right angle,
add to each the angle ABC,
then the whole angle DBA *is equal to the whole* FBC. (ax. 2)
Hence, in the two triangles ABD, FBC,
because the two sides AB, BD, are equal to the two FB, BC, each to each, (def. 30)
and the angle DBA equal to the angle FBC;
therefore the base AD is equal to the base FC,
and the triangle ABD *to the triangle* FBC. (I. 4)

Again, because the parallelogram BL and the triangle ABD are upon the same base BD, and between the same parallels BD, AL;
therefore the parallelogram BL *is double of the triangle* ABD; (I. 41)
and because the square GB and the triangle FBC are upon the same base FB, and between the same parallels FB, GC;
therefore the square GB *is double of the triangle* FBC;
but the doubles of equals are equal to one another; (ax. 6)
therefore the parallelogram BL *is equal to the square* GB.

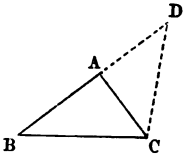
In the same manner, by joining AE, BK, it is proved that the parallelogram CL is equal to the square HC.
Therefore the whole square BDEC *is equal to the two squares* GB, HC: (ax. 2.)
and the square BDEC is described upon the straight line BC,
and the squares GB, HC, upon BA, AC;
wherefore the square upon the side BC, is equal to the squares upon the sides BA, AC.
Therefore, in any right-angled triangle, &c. Q. E. D.

PROP. XLVIII.—THEOREM.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it;
then the angle contained by these two sides is a right angle.

(References—Prop. I. 3, 8, 11, 47; ax. 1, 2.)

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other sides BA, AC.
Then the angle BAC shall be a right angle.



CONSTRUCTION

From the point A draw AD at right angles to AC, (I. 11) and make AD equal to BA, (I. 3) and join DC.

DEMONSTRATION

Then, because DA is equal to AB,

the square of DA *is equal to the square of* AB;

to each of these equals add the square of AC,

therefore the squares of DA, AC, *are equal to the squares of* BA, AC (ax. 2)

but because DAC is a right angle, (constr.)

herefore the square of DC *is equal to the squares of* DA, AC; (I. 47)

and the square of BC is equal to the squares of BA, AC; (hyp.)

therefore the square of DC is equal to the square of BC; (ax. 1)

wherefore also the side DC *is equal to the side* BC.

Hence, in the two triangles DAC, BAC,
because the side DA is equal to the side AB, (constr.)
and AC common to both triangles,
the two DA, AC, are equal to the two BA, AC, each to each,
and the base DC has been proved to be equal to the base BC;

therefore the angle DAC *is equal to the angle* BAC; (I. 8)

but DAC is a right angle; (constr.)

therefore also BAC is a right angle. (ax. 1.)

Therefore, if the square, &c. Q. E. D.

BOOK II.

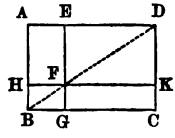
DEFINITIONS.

I.

EVERY right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.



'Thus the parallelogram HG, together with the complements AF, FC, is the Gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the Gnomon.'

PROP. I.—THEOREM.

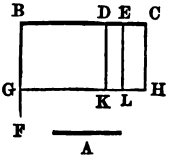
If there be two straight lines, one of which is divided into any number of parts;

then the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

(References—Prop. I. 3, 11, 31, 34.)

Let A and BC be two straight lines; and let BC be divided into any number of parts in D and E.

Then the rectangle contained by the straight lines A and BC is equal to the rectangle contained by A and BD, together with that contained by A and DE, and A and EC.



CONSTRUCTION

From the point B, draw BF at right angles to BC, (I. 11)
and make BG equal to A; (I. 3)
through G draw GH parallel to BC; (I. 31)
and through the points D, E, C, draw DK, EL, CH, parallel to BG.

DEMONSTRATION

Then the rectangle BH is equal to the rectangles BK, DL, EH;
but because BH is contained by GB, BC,
and GB is equal to A; (constr.)
therefore BH *is contained by* A *and* BC,
and, because BK is contained by GB, BD, of which GB is equal to A,
therefore BK *is contained by* A *and* BD;
and because DL is contained by DK and DE,
and DK, that is, BG, (I. 34) is equal to A;
therefore DL *is contained by* A *and* DE;
and in like manner EH is contained by A and EC;

therefore the rectangle contained by A and BC is equal to the several rectangles contained by A and BD, by A and DE, and by A and EC.

Wherefore, if there be two straight lines, &c. Q. E. D.

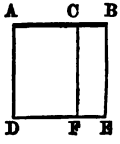
PROP. II.—THEOREM.

If a straight line be divided into any two parts;
then the rectangles contained by the whole and each of the parts are together equal to the square of the whole line.

(References — Prop. I. 31, 46.)

Let AB be divided into any two parts in C.

Then the rectangle* contained by AB and BC, together with the rectangle contained by AB and AC, shall be equal to the square of AB.



CONSTRUCTION

Upon AB describe the square ADEB, (I. 46)
and through C draw CF parallel to AD or BE. (I. 31.)

DEMONSTRATION

Then AE is equal to the rectangles AF, CE.
But AE is the square of AB;

therefore the square of AB *is equal to the rectangles* AF, CE.

But AF is contained by DA, AC, of which AD is equal to AB, (def. 30)

therefore AF *is the rectangle contained by* AB, AC;

and CE is contained by AB, BC, since BE is equal to AB;

therefore the rectangle contained by AB, AC, together with the rectangle AB, BC, is equal to the square of AB.

If, therefore, a straight line, &c. Q. E. D.

PROP. III.—THEOREM.

If a straight line be divided into any two parts;
then the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

(References — Prop. I. 31, 46.)

Let the straight line AB be divided into any two parts in the point C.

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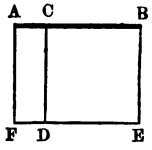
* N.B. To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines, AB and AC, is sometimes simply called the rectangle *AB, AC.*

Then the rectangle AB, BC, shall be equal to the rectangle AC, CB, together with the square of CB.



CONSTRUCTION

Upon BC describe the square CDEB; (I. 46)
produce ED to F; and through A draw AF parallel to CD or BE. (I. 31.)

DEMONSTRATION

Then the rectangle AE is equal to the rectangles AD, CE.
But AE is the rectangle contained by AB, BE, of which BE is equal to BC, (def. 30)

therefore AE *is the rectangle contained by* AB, BC;

and AD is contained by AC, CD, of which CD is equal to BC,

therefore AD *is contained by* AC, CB;

and CE is the square of BC;

therefore the rectangle AB, BC, is equal to the rectangle AC, CB, together with the square of BC.

If, therefore, a straight line be divided, &c. Q. E. D.

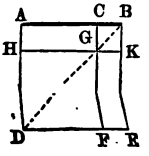
PROP. IV.—THEOREM.

If a straight line be divided into any two parts;
then the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

(References—Prop. I. 5, 6, 29, 31, 34, 43, 46.)

Let the straight line AB be divided into any two parts in C.

Then the square of AB shall be equal to the squares of AC, CB, together with twice the rectangle AC, CB.



CONSTRUCTION

Upon AB describe the square ADEB, (I. 46) and join DB;
through C draw CGF parallel to BE or AD,
and through G draw HGK parallel to AB or DE. (I. 31.)

DEMONSTRATION

Then, because CF is parallel to AD, and BD falls upon them,

therefore the exterior angle BGC *is equal to the interior and opposite angle* ADB; (I. 29)

but, because AB is equal to AD, being the sides of the square ADEB; (def. 30)

therefore, the angle ADB is equal to the angle CBG; (I. 5)

and the angle CGB is therefore equal to the angle CBG; (ax. 1)

wherefore the side BC *is equal to the side* CG; (I. 6)

but CB is equal also to GK, and CG to BK; (I. 34)

therefore the figure CGKB *is equilateral.* (ax. 1.)

It is likewise rectangular,

for since CG is parallel to BK, and CB meets them,

therefore the angles KBC, GCB, *are equal to two right angles*; (I. 29)

but KBC is a right angle; (def. 30)

wherefore GCB is a right angle,

and therefore also the opposite angles CGK, GKB, *are right angles,* (I. 34)

wherefore the figure CGKB *is rectangular;*

but it is also equilateral, as was demonstrated,

therefore CGKB *is a square;*

and it is upon the side CB.

For the same reason, also,

HF *is a square,*

and it is upon the side HG, which is equal to AC; (I. 34)

therefore HF, CK, *are the squares of* AC, CB.

And, because the complement AG is equal to the complement GE, (I. 43)

and that AG is the rectangle contained by AC, CB, since GC is equal to CB; (def. 30)

therefore also GE is equal to the rectangle AC, CB;

wherefore AG, GE, *are equal to twice* the rectangle AC, CB;

and HF, CK, are the squares of AC, CB;

therefore the four figures HF, CK, AG, GE, *are equal to the squares of* AC, CB, *together with twice the rectangle* AC, CB;

but HF, CK, AG, GE, make up the whole figure ADEB, which is the square of AB;

therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB.

Wherefore, if a straight line, &c. Q. E. D.

Cor. From the demonstration, it is manifest that the parallelograms about the diameter of a square are likewise squares.

PROP. V.—THEOREM.

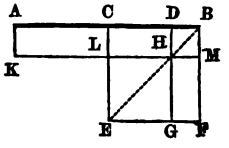
If a straight line be divided into two equal parts, and also into two unequal parts;

then the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

(References—Prop. I. 31, 34, 36, 43, 46; II. 4, Cor.)

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D.

Then the rectangle AD, DB, together with the square of CD, shall be equal to the square of CB.



CONSTRUCTION

Upon CB describe the square CEFB, (I. 46) and join BE;
through D draw DHG parallel to CE or BF; (I. 31)
through H draw KLM parallel to CB or EF;
and through A draw AK parallel to CL or BM.

DEMONSTRATION

Then, because the complement CH is equal to the complement HF, *(I. 43)*
to each of these add DM;

therefore the whole CM *is equal to the whole* DF;

but CM is equal to AL, (I. 36) because AC is equal to CB; (hyp.)

therefore also, AL *is equal to* DF;

to each of these equals add CH,

therefore the whole AH *is equal to* DF *and* CH.

But AH is contained by AD, DB, since DH is equal to DB;
and DF, together with CH, is the gnomon CMG;

therefore the gnomon CMG *is equal to the rectangle* AD, DB; (ax. 1)

to each of these equals add LG, which is equal to the square of CD, (II. 4, Cor.) since CD is equal to LH;

therefore the gnomon CMG, *together with* LG, *is equal to the rectangle* AD, DB, *together with the square of* CD;

but the gnomon CMG and LG make up the whole figure CEFB, which is the square of CB;

therefore the rectangle AD, DB, together with the square of CD, is equal to the square of CB.

Wherefore, if a straight line, &c. Q. E. D.

Cor. From this proposition it is manifest that the difference of the squares of two unequal lines AC, CD, is equal to the rectangle contained by their sum and difference.

PROP. VI.—THEOREM.

If a straight line be bisected, and produced to any point;

then the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

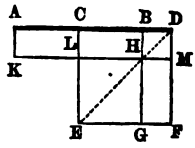
(References—Prop. I. 31, 36, 43; II. 4, Cor.)

Let the straight line AB be bisected in C, and produced to D.

Then the rectangle AD, DB, together with the square of BC, shall be equal to the square of CD.

CONSTRUCTION

Upon CD describe the square CEFD, (I. 46) and join DE;
through B draw BHG parallel to CE or DF; (I. 31)
through H draw KLM parallel to AD or EF;
and through A draw AK parallel to CL or DM.



DEMONSTRATION

Because AC is equal to CB, (hyp.)

therefore the rectangle AL *is equal to* CH, (I. 36)

but CH is equal to HF; (I. 43)

therefore also AL *is equal to* HF;

to each of these add CM;

therefore the whole AM *is equal to the gnomon* CMG.

But AM is the rectangle contained by AD, DB, since DM is equal to DB; (II. 4, Cor.)

therefore the gnomon CMG *is equal to the rectangle* AD, DB;

add to each of these LG, which is equal to the square of CB;

therefore the rectangle AD, DB, *together with the square of* CB, *is equal to the gnomon* CMG, *and the figure* LG;

but the gnomon CMG and LG make up the whole figure CEFD, which is the square of CD;

therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD.

Wherefore, if a straight line, &c. Q. E. D.

PROP. VII.—THEOREM.

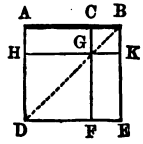
If a straight line be divided into any two parts;

then the squares of the whole line and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

(References — Prop. I. 34, 43, 46; II. 4. Cor.)

Let the straight line AB be divided into any two parts in the point C,

***Then* the squares of AB, BC, shall be equal to twice the rectangle AB, BC, together with the square of AC.**



CONSTRUCTION

Upon AB describe the square ADEB, (I. 46) and join BD;
through C draw CGF parallel to AD or BE, (I. 31)
and through G draw HGK parallel to AB or DE.

DEMONSTRATION

Then, because AG is equal to GE, (I. 43)
add to each of them CK;
therefore the whole AK is equal to the whole CE;
and therefore AK, CE, *are double of* AK;
but AK, CE, are the gnomon AKF, together with the square CK,
therefore the gnomon AKF, *together with the square* CK, *is double of* AK;
but twice the rectangle AB, BC, is double of AK, for BK is equal to BC; (II. 4, Cor.)
therefore the gnomon AKF, *together with the square* CK, *is equal to twice the rectangle* AB, BC;
to each of these equals add HF, which is equal to the square of AC;
therefore the gnomon AKF, *together with the squares* CK, HF, *is equal to twice the rectangle* AB, BC, *and the square of* AC;
but the gnomon AKF, together with the squares CK, HF, make up the whole figures ADEB and CK, which are the squares of AB and BC;
therefore the squares of AB and BC are equal to twice the rectangle AB, BC, together with the square of AC.

Wherefore, if a straight line, &c. Q. E. D.

PROP. VIII.—THEOREM.

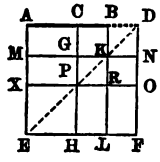
If a straight line be divided into any two parts;
then four times the rectangle contained by the whole line, and one of the

parts, together with the square of the other part, is equal to the square of the straight line which is made up of the whole and that part.

(References — Prop. I. 3, 31, 34, 36, 43, 46 ; II. 4, Cor.)

Let the straight line AB be divided into any two parts in the point C.

Then four times the rectangle AB, BC, together with the square of AC, shall be equal to the square of the straight line made up of AB and BC together.



CONSTRUCTION

Produce AB to D, so that BD be equal to CB ; (I. 3)
upon AD describe the square AEFD ; (I. 46)
and construct two figures such as in the preceding.

DEMONSTRATION

Then, because CB is equal to BD, (constr.)
and that CB is equal to GK, and BD to KN ; (I. 34)
therefore GK *is equal to* KN ;
for the same reason, PR is equal to RO ;
and because CB is equal to BD, and GK to KN,
therefore the rectangle CK *is equal to* BN, *and* GR *to* RN ; (I. 36)
but CK is equal to RN, because they are the complements of the parallelogram CO ; (I. 43)
therefore also BN is equal to GR,
and therefore the four rectangles BN, CK, GR, RN, *are equal to one another, and so are quadruple of one of them,* CK.

Again, because CB is equal to BD,
and that BD is equal to BK, that is CG ; (I. 34)
and because CB is equal to GK, that is GP ;
therefore CG *is equal to* GP ;
and because CG is equal to GP, and PR to RO,
the rectangle AG *is equal to* MP, *and* PL to RF ;

but MP is equal to PL, because they are the complements of the parallelogram ML; (I. 43)

and AG is equal to RF;

therefore the four rectangles AG, MP, PL, RF, *are equal to one another, and so are quadruple of one of them,* AG.

And it was demonstrated that the four CK, BN, GR, and RN, are quadruple of CK;

therefore the eight rectangles which contain the gnomon AOH, *are quadruple of* AK.

And because AK is the rectangle contained by AB, BC, for BK is equal to BC,

therefore four times the rectangle AB, BC, *is quadruple of* AK;

but the gnomon AOH was demonstrated to be quadruple of AK;

therefore four times the rectangle AB, BC, *is equal to the gnomon* AOH;

to each of these add XH, which is equal to the square of AC; (II. 4, Cor.)

therefore four times the rectangle AB, BC, *together with the square of* AC, *is equal to the gnomon* AOH, *and the square* XH;

but the gnomon AOH and XH make up the whole figure AEFD, which is the square of AD;

therefore four times the rectangle AB, BC, together with the square of AC, is equal to the square of AD,

that is, of AB and BC added together in one straight line.

Wherefore, if a straight line, &c. Q. E. D.

PROP. IX.—THEOREM.

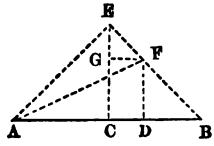
If a straight line be divided into two equal and also into two unequal parts;

then the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

(Reference—Prop. I. 3, 5, 6, 11, 29, 31, 32, 34, 47.)

Let the straight line AB be divided into two equal parts in the point C, and *into* unequal parts in the point D.

Then the squares of AD, BD, shall be together double of the squares of AC, CD.



CONSTRUCTION

From the point C draw CE at right angles to AB. (I. 11)
and make it equal to AC or CB, (I. 3) and join EA, EB;
through D draw DF parallel to CE; (I. 31) meeting GB in F;
and through F draw FG parallel to AB; and join AF.

DEMONSTRATION

Then, because AC is equal to CE, (constr.)

the angle EAC *is equal to the angle* AEC; (I. 5)

and because the angle ACE is a right angle,

the angles AEC, EAC, *together make one right angle*; (I. 32)

and they are equal to one another;

therefore each of the angles AEC, EAC, *is half a right angle.*

For the same reason each of the angles CEB, EBC, is half a right angle;

therefore the whole AEB *is a right angle.*

And, because the angle GEF is half a right angle,
and EGF a right angle, for it is equal to the interior and opposite angle ECB, (I. 29)
the remaining angle EFG is half a right angle;
therefore the angle GEF is equal to the angle EFG,
wherefore the side EG *is equal to the side* GF. (I. 6.)

Again, because the angle at B is half a right angle,
and FDB a right angle, for it is equal to the interior and opposite angle ECB, (I. 29)
the remaining angle BFD is half a right angle;
therefore the angle at B is equal to the angle BFD,
wherefore the side DF *is equal to the side* DB. (I. 6.)

And because AC is equal to CE,
the square of AC is equal to the square of CE,

therefore the squares of AC, CE, *are double of the squares of* AC;

but because ACE is a right angle

the square of EA is equal to the squares of AC, CE, (I. 47)

therefore the square of EA *is double of the square of* AC.

Again, because EG is equal to GF,

the square of EG is equal to the square of GF,

therefore the squares of EG, GF, *are double of the square of* GF

but the square of EF is equal to the squares of EG, GF; (I. 47)

therefore the square of EF *is double of the square of* GF;

and GF is equal to CD; (I. 34)

therefore the square of EF *is double of the square of* CD;

but the square of EA is likewise double of the square of AC;

therefore the squares of EA, EF, *are double of the squares of* AC, CD.

And because AEF is a right angle,

the square of AF is equal to the squares of AE, EF, (I. 47)

therefore the square of AF *is double of the squares of* AC, CD;

but the squares of AD, DF, are equal to the squares of AF, because ADF is a right angle, (I. 47)

therefore the squares of AD, DF, *are double of the squares of* AC, CD;

and DF is equal to DB;

therefore the squares of AD, DB, are double of the squares of AC, CD.

Therefore if a straight line, &c. Q. E. D.

PROP. X.—THEOREM.

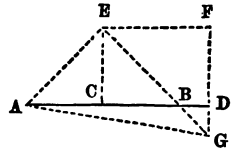
If a straight line be bisected, and produced to any point;

then the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

(References—Prop. I. 5, 6, 11, 15, 29, 31, 32, 34, 46 Cor. 47.)

Let the straight line AB be bisected in C, and produced to the point D.

Then the squares of AD, DB, shall be double of the squares of AC, CD.



CONSTRUCTION

From the point C draw CE at right angles to AB; (I. 11)
make it equal to AC or CB, (I. 3) and join AE, EB;
through E draw EF parallel to AB, (I. 31)
and through D draw DF parallel to CE.
Then because the straight line EF meets the parallels EC, FD, the angles CEF, EFD, are equal to two right angles; (I. 29)
therefore the angles BEF, EFD, *are less than two right angles;*
but 'straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet if produced far enough;' (ax. 12)
therefore EB, FD, *will meet, if produced towards* B *and* D;
let them meet in G, and join AG.

DEMONSTRATION

Then, because AC is equal to CE,
the angle CEA *is equal to the angle* EAC; (I. 5)
and the angle ACE is a right angle,
therefore each of the angles CEA, EAC, *is half a right angle.* (I. 32.)
For the same reason each of the angles CEB, EBC, is half a right angle;
therefore the whole AEB *is a right angle.*

And because EBC is half a right angle, the angle DBG which is vertically opposite is also half a right angle, (I. 15)
but BDG is a right angle, because it is equal to the alternate angle DCE; (I. 29)
therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG;
wherefore also the side BD *is equal to the side* DG. (I. 6.)
Again, because EGF is half a right angle,
and that the angle at F is a right angle, because it is equal to the *opposite angle ECD*, (I. 34)
the angle FEG is half a right angle,

and therefore equal to the angle EGF;
wherefore also the side GF *is equal to the side* FE.

And because EC is equal to CA,
the square of EC is equal to the square of CA;
therefore the squares of EC, CA, *are double of the square of* CA;
but the square of EA is equal to the squares of EC, CA; (I. 47)
therefore the square of EA *is double of the square of* AC.

Again, because GF is equal to EF,
the square of GF is equal to the square of EF;
and therefore the squares of GF, FE, *are double of the square of* EF;
but the square of EG is equal to the squares of GF, EF; (I. 47)
therefore the square of EG *is double of the square of* EF;
and EF is equal to CD; (I. 34)
wherefore the square of EG *is double of the square of* CD;

But it was demonstrated that the square of EA is double of the square of AC;
therefore the squares of AE, EG, *are double of the squares of* AC, CD;
but the square of AG is equal to the squares of AE, EG; (I. 47)
therefore the square of AG *is double of the squares of* AC, CD;
but the square of AG is also equal to the squares of AD, DG; (I. 47)
therefore the squares of AD, DG, *are double to the squares of* AC, CD;
but DG is equal to DB;
therefore the squares of AD, DB, are double of the squares of AC, CD.
Wherefore, if a straight line, &c. Q. E. D.

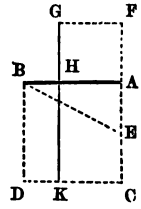
PROP. XI.—PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

(References — Prop. I. 3, 10, 46, 47; II. 6.)

Let AB be the given straight line.
It is *required* to divide AB into two parts, so that the rectangle

contained by the whole and one of the parts shall be equal to the square of the other part.



CONSTRUCTION

Upon AB describe the square ABDC; (I. 46)
bisect AC in E, (I. 10) and join BE;
produce CA to F, and make EF equal to EB; (I. 3)
and upon AF describe the square FGHA. (I. 46.)

Then AB shall be divided in H, so that the rectangle AB, BH, is equal to the square of AH.

Produce GH to K.

DEMONSTRATION

Then, because the straight line AC is bisected in E, and produced to the point F,

the rectangle CF, FA, *together with the square of* AE, *is equal to the square of* EF; (II. 6)

but EF is equal to EB;

therefore the rectangle CF, FA, *together with the square of* AE, *is equal to the square of* EB;

and the squares of BA, AE, are equal to the square of EB, (I. 47) because the angle EAB is a right angle;

therefore the rectangle CF, FA, *together with the square of* AE, *is equal to the squares of* BA, AE;

take away the square of AE, which is common to both;

therefore the remaining rectangle CF, FA, *is equal to the square of* AB.

But the figure FK is the rectangle contained by CF, FA, since AF is equal to FG; (def. 30)
and AD is the square of AB;

therefore the figure FK *is equal to* AD;

take away the common part AK,

and the remainder FH *is equal to* the remainder HD

but HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the square of AH;

therefore the rectangle AB, BH, is equal to the square of AH.

Wherefore the straight line AB is divided in H, so that the rectangle AB, BH, is equal to the square of AH. Q. E. F.

PROP. XII.— THEOREM.

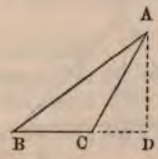
In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced;

then the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

(References — Prop. I. 47; II. 4.)

Let ABC be an obtuse-angle triangle, having the obtuse angle ACB; and from the point A let AD be drawn perpendicular to BC produced. (I. 12.)

Then the square of AB shall be greater than the squares of AC, CB, by twice the rectangle BC, CD.



DEMONSTRATION

Because the straight line BD is divided into two parts in the point C,

the square of BD *is equal to the squares of* BC, CD, *and twice the rectangle* BC, CD; (II. 4)

to each of these equals add the square of DA;

then the squares of BD, DA, *are equal to the squares of* BC, CD, DA, *and twice the rectangles* BC, CD;

but the square of BA is equal to the squares of BD, DA, (I. 47) because *the angle at D is* a right angle;

and the square of CA is equal to the squares of CD, DA; (I. 47)

therefore the square of BA *is equal to the squares of* BC, CA, *and twice the rectangle* BC, CD;

that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD.

Wherefore, in obtuse-angled triangles, &c. Q. E. D.

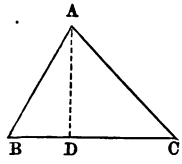
PROP. XIII. — THEOREM.

In every triangle, the square of the side subtending either of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of those sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

(References — Prop. I. 16, 47; II. 3, 7, 12.)

Let ABC be any triangle, and the angle at B one of its acute angles; and upon BC, one of the sides containing it, let fall the perpendicular AD from the opposite angle. (I. 12.)

Then the square of AC opposite to the angle B, shall be less than the squares of CB, BA, by twice the rectangle CB, BD.



DEMONSTRATION

First, Let AD fall within the triangle ABC.

Then, because the straight line CB is divided into two parts in the point D,

the squares of CB, BD, *are equal to twice the rectangle contained by* CB, BD, *and the square of* DC; (II. 7)

to each of these equals add the square of AD;

therefore the squares of CB, BD, DA, *are equal to twice the rectangle CB, BD, and the squares of* AD, DC;

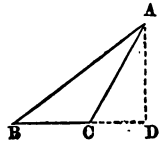
but the square of AB is equal to the squares of BD, DA, (I. 47) because the angle BDA is a right angle;

and the square of AC is equal to the squares of AD, DC;

therefore the squares of CB, BA, *are equal to the square of* AC, *and twice the rectangle* CB, BD;

that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB, BD.

Secondly, Let AD fall without the triangle ABC.



Then, because the angle at D is a right angle,

the angle ACB is greater than a right angle; (I. 16)

and therefore the square of AB *is equal to the squares of* AC, CB, *and twice the rectangle* BC, CD; (II. 12)

to each of these equals add the square of BC,

then the squares of AB, BC, *are equal to the square of* AC, *and twice the square of* BC, *and twice the rectangle* BC, CD;

but because BD is divided into two parts in C,

the rectangle DB, BC, *is equal to the rectangle* BC, CD, *and the square of* BC; (II. 3)

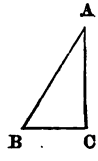
and the doubles of these are equal;

that is, twice the rectangle DB, BC, is equal to twice the rectangle BC, CD, and twice the square of BC;

therefore the squares of AB, BC, *are equal to the square of* AC, *and twice the rectangle* DB, BC;

wherefore the square of AC alone is less than the squares of AB, BC, by twice the rectangle DB, BC.

Lastly, let the side AC be perpendicular to BC.



DEMONSTRATION

Then BC is the straight line between the perpendicular and the acute angle at B;

and it is manifest that the squares of AB, BC, are equal to the square of AC, and twice the square of BC.

Therefore, in every triangle, &c. Q. E. D.

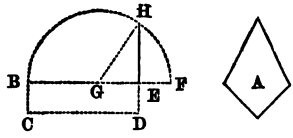
PROP. XIV.—PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

(References—Prop. I. 3, 10, 45, 47; II. 5.)

Let A be the given rectilineal figure.

It is required to describe a square that shall be equal to A.



CONSTRUCTION

Describe the rectangular parallelogram BCDE equal to the rectilineal figure A. (I. 45.)

If, then, the sides of it, BE, ED, are equal to one another, it is a square, and what was required is now done.

But if they are not equal, produce one of them, BE to F, and make EF equal to ED;

bisect BF in G, (I. 10) and from the centre G, at the distance GB, or GF, describe the semicircle BHF,

produce DE to meet the circumference in H.

Then the square described upon EH shall be equal to the given rectilineal figure A.

Join GH.

DEMONSTRATION

Because the straight line BF is divided into two equal parts in the point *G, and into* two unequal parts in E,

the rectangle BE, EF, *together with the square of* EG, *is equal to the square of* GF; (II. 5)

but GF is equal to GH; (def. 15)

therefore the rectangle BE, EF, *together with the square of* EG, *is equal to the square of* GH;

but the squares of HE, EG, are equal to the square of GH; (I. 47)

therefore the rectangle BE, EF, *together with the square of* EG, *is equal to the squares of* HE, EG;

take away the square of EG, which is common to both,

and the remaining rectangle BE, EF, *is equal to the square of* EH.

But the rectangle contained by BE, EF, is the parallelogram BD, because EF is equal to ED;

therefore BD *is equal to the square of* EH

but BD is equal to the rectilineal figure A;

therefore the rectilineal figure A is equal to the square of EH.

Wherefore a square has been made equal to the given rectilineal figure A — viz., the square described upon EH. Q. E. F.

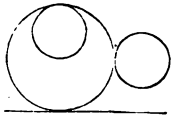
BOOK III.

DEFINITIONS.

I.

EQUAL circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumference are equal.

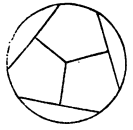
II.



A straight line is said to touch a circle when it meets the circle, and being produced does not cut it.

III.

Circles are said to touch each other which meet, but do not cut each other.



IV.

Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal

V.

And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

VI.

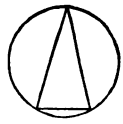
A segment of a circle is a figure contained by a straight line, and the circumference which it cuts off.

VII.

The angle of a segment is that which is contained by the straight line and the circumference.

VIII.

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.

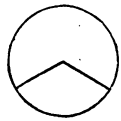


IX.

An angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.

X.

A sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



XI.

Similar segments of circles are those in which the angles are equal, or which contain equal angles.

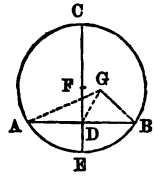
PROP. I.—PROBLEM.

To find the centre of a given circle.

(References—Prop. I. 8, 10, 11.)

Let ABC be the given circle.

It is required to find its centre.



CONSTRUCTION

Draw within the circle any straight line AB, and bisect AB in D; (I. 10)

from the point D draw DC at right angles to AB (I. 11), produce it to E, and bisect CE in F.

Then the point F shall be the centre of the circle ABC.

DEMONSTRATION

For, if it be not, let, if possible, G be the centre, and join GA, GD, GB.

Then, because DA is equal to DB, (constr.) and DG common to the two triangles ADG, BDG,

the two sides AD, DG, are equal to the two BD, DG, each to each;

and the base GA is assumed to be equal to the base GB, (I. def. 15) because they are drawn from the centre G;

therefore the angle ADG *must be equal to the angle* GDB; (I. 8)

but 'when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle;' (I. def. 10)

therefore the angle GDB *must be a right angle;*

but FDB is a right angle; (constr.)

wherefore the angle FDB *must be equal to the angle* GDB,

the greater to the less, which is impossible;

therefore G *is not the centre of the circle* ABC.

In the same manner it can be shown, that no other point but F is the *centre;*

therefore F is the centre of the circle ABC,

which was to be found.

Cor.—From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

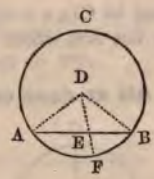
PROP. II.—Theorem.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

(References—Prop. I. 5, 16, 19; III. 1.)

Let ABC be a circle, and A, B, any two points in the circumference.

Then the straight line drawn from A to B shall fall within the circle.



CONSTRUCTION

Find D the centre of the circle ABC, (III. 1) and join AD, DB; in the circumference AB take any point F;
join DF, and let it cut the straight line AB in E.

DEMONSTRATION

Because DA is equal to DB, (I. def. 15)
the angle DAB *is equal to the angle* DBA; (I. 5)
and because AE, a side of the triangle DAE, is produced to B,
the exterior angle DEB *is greater than the interior and opposite angle* DAE; (I. 16)
but the angle DAE was proved to be equal to the angle DBE;
therefore the angle DEB *is also greater than the angle* DBE;
but to the greater angle the greater side is opposite; (I. 19)
therefore DB *is greater than* DE;
but DB is equal to DF; (I. def. 15)

therefore DF is greater than DE,

wherefore the point E *lies within the circle.*

In the same manner it may be proved that every point in AB lies within the circle;

therefore the straight line AB lies within the circle.

Wherefore, if any two points, &c. Q. E. D.

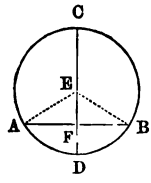
PROP. III.—THEOREM.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and conversely, if it cut it at right angles, it shall bisect it.

(References—Prop. I. 5, 8, 26; III. 1.)

Let ABC be a circle, and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, in the point F.

Then CD shall cut AB at right angles.



CONSTRUCTION

Take E the centre of the circle (III. 1), and join EA, EB.

DEMONSTRATION

Then, because AF is equal to FB (hyp.), and FE common to the two triangles AFE, BFE,

there are two sides in the one equal to two sides in the other, each to each;

and the base EA is equal to the base EB; (I. def. 15)

therefore the angle AFE *is equal to the angle* BFE; (I. 8)

but 'when a straight line standing upon another straight line makes the *adjacent angles* equal to one another, each of them is a right angle;' *(I. def. 10)*

therefore each of the angles AFE, BFE, *is a right angle*;

wherefore the straight line CD, drawn through the centre, bisecting another AB that does not pass through the centre, cuts the same at right angles.

But let CD cut AB at right angles.

Then CD shall also bisect it, that is, AF shall be equal to FB.

The same construction being made;

because EA, EB, from the centre are equal to one another,

the angle EAF *is equal to the angle* EBF; (I. 5)

and the right angle AFE is equal to the right angle BFE; (I. def. 10)

hence, in the two triangles EAF, EBF,

there are two angles in the one equal to two angles in the other, each to each;

and the side EF, which is opposite to one of the equal angles in each, is common to both;

therefore the other sides are equal; (I. 26)

therefore AF is equal to FB.

Wherefore, if a straight line, &c. Q. E. D.

PROP. IV.—THEOREM.

If in a circle two straight lines cut one another which do not both pass through the centre, they do not bisect each other.

(References — Prop. III. 1, 3.)

Let ABCD be a circle, and AC, BD, two straight lines in it which cut one another in the point E, and do not both pass through the centre.

Then AC, BD, shall not bisect one another.

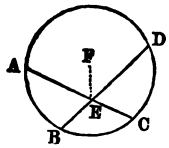
CONSTRUCTION

For, if it be possible, let AE be equal to EC, and BE to ED.

If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre;

but if neither of them pass through the centre,

take F the centre of the circle, (III. 1) and join EF.



DEMONSTRATION

Because FE, a straight line through the centre, is assumed to bisect another AC which does not pass through the centre,
therefore it must cut it at right angles; (III. 3)
wherefore FEA *must be a right angle.*

Again, because the straight line FE is assumed to bisect the straight line BD which does not pass through the centre,
therefore it must cut it at right angles; (III. 3)
wherefore FEB *must be a right angle;*

but FEA was shown to be a right angl
therefore FEA *must be equal to the angle* FEB,
the less to the greater, which is impossible;
therefore AC, BD, do not bisect one another.

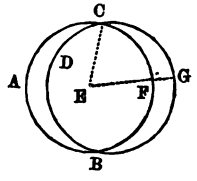
Wherefore, if in a circle, &c. Q. E. D.

PROP. V.—THEOREM.

If two circles cut one another, they shall not have the same centre.

(References — I. def. 15.)

Let the two circles ABC, CDG, cut one another in the points B, C.
Then they shall not have the same centre.



CONSTRUCTION

For, if it be possible, let E be their centre; join EC, and draw any *straight line* EFG meeting them in F and G.

DEMONSTRATION

And because E is assumed to be the centre of the circle ABC,

EC *must be equal to* EF; (I. def. 15)

again, because E is assumed to be the centre of the circle CDG,

EC *must be equal to* EG;

but EC was shown to be equal to EF;

therefore EF *must be equal to* EG;

the less to the greater, which is impossible;

therefore E is not the centre of the circles ABC, CDG.

Wherefore, if two circles, &c. Q. E. D.

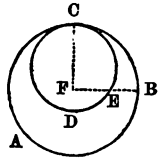
PROP. VI.—THEOREM.

If one circle touch another internally, they shall not have the same centre.

(References — I. def. 15.)

Let the circle CDE, touch the circles ABC internally in the point C.

Then they shall not have the same centre.



CONSTRUCTION

For, if they can, let the centre be F; join FC, and draw any straight line FEB meeting them in E and B.

DEMONSTRATION

Because F is assumed to be the centre of the circle ABC,

therefore CF *must be equal to* FB; (I. def. 15)

also, because F is assumed to be the centre of the circle CDE,

therefore CF *must be equal to* FE;

but CF was shown to be equal to FB

therefore FE *must be equal to* FB; (AX. I.)
the less to the greater, which is impossible;

wherefore F is not the centre of the circles ABC, CDE.

Therefore, if two circles, &c. Q. E. D.

PROP. VII.— THEOREM.

If any point be taken in the diameter of a circle, which is not the centre;

then of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote; and from the same point there can be drawn only two straight lines to the circumference that are equal to each other, one upon each side of the shortest line.

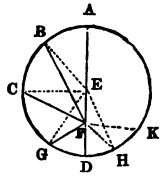
(References — Prop. I. 4, 20, 23, 24.)

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre; let the centre be E.

Then of all the straight lines FB, FC, FG, &c., that can be drawn from F to the circumference, FA which passes through the centre, shall be the greatest;

and FD, the other part of the diameter AD, shall be the least;

and of the others, FB, the nearer to FA, shall be greater than FC, the more remote; and FC greater than FG.



CONSTRUCTION

Join BE, CE, GE.

DEMONSTRATION

Because two sides of a triangle are greater than the third,
therefore BE, EF, *are greater than* BF; (I. 20)
but AE is equal to EB; (I. def. 15)
therefore AE, EF, *that is* AF, *is greater than* BF.

Again, in the triangles BEF, CEF,
because BE is equal to CE, and EF common to both,
the two sides BE, EF, are equal to the two CE, EF, each to each;
but the angle BEF is greater than the angle CEF; (ax. 9)
therefore the base BF *is greater than the base* CF; (I. 24)
for the same reason, CF is greater than GF.

Again, because GF, FE, are greater than EG, (I. 20) and EG is equal to ED;
therefore GF, FE, *are greater than* ED;
take away the common part FE,
and the remainder GF *is greater than the remainder* FD; (ax. 5)
therefore FA is the greatest of all the straight lines from F to the circumference,
and FD is the least;
and BF is greater than CF, and CF than GF.

Also there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the shortest line FD.

CONSTRUCTION

At the point E, in the straight line EF, make the angle FEH equal to the angle GEF, (I. 23) and join FH.

DEMONSTRATION

Then in the triangles GEF, HEF,
because GE is equal to EH, (I. def. 15) and EF common to both,
the two sides GE, EF, are equal to the two HE, EF, each to each;
and the angle GEF is equal to the angle HEF; (constr.)
therefore the base FG *is equal to the base* FH; (I. 4)
but besides FH, no straight line can be drawn from F to the circumference equal to FG;
for, if there can, let it be FK;
and because FK is assumed to be equal to FG, and FG is equal to FH,
therefore FK must be equal to FH; (ax. 1)

that is, a line nearer to that which passes through the centre, is equal to one which is more remote;
which has been proved to be impossible.
Therefore, if any point be taken, &c. Q. E. D.

PROP. VIII.— THEOREM.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre;

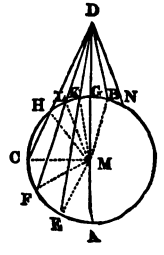
then of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre, is always greater than one more remote; but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest that which is nearer to the least is always less than one more remote; and only two equal straight lines can be drawn from the same point to the circumference, one upon each side of the line which passes through the centre.

(References — Prop. I. 4, 20, 21, 23, 24; III. 1.)

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DC, be drawn to the circumference whereof DA passes through the centre.

Then, of those which fall upon the concave part of the circumference AEFC, AD which passes through the centre shall be the greatest; and the nearer to it is always greater than the more remote, viz., DE than DF, and DF than DC.

but of those which fall upon the convex circumference HLKG, the least shall be DG between the point D and the diameter AG; and the nearer to it is always less than the more remote, viz., DK than DL, and DL than DH.



CONSTRUCTION

Take M the centre of the circle ABC, (III. 1) and join ME, MF, MC, MK, ML, MH.

DEMONSTRATION

And because AM is equal to EM,
add MD to each,
therefore AD *is equal to* EM, MD ; (ax. 2)
but EM, MD, are greater than ED ; (I. 20)
therefore also AD *is greater than* ED.

Again, in the triangles EMD, FMD,
because ME is equal to MF, and MD common to both,
EM, MD, are equal to FM, MD, each to each,
but the angle EMD is greater than the angle FMD ; (ax. 9)
therefore the base ED *is greater than the base* FD. (I. 24.)
In like manner it may be shown that FD is greater than CD ;
therefore DA is the greatest line,
and DE is greater than DF, and DF than DC

And because MK, KD, are greater than MD, (I. 20)
and MK is equal to MG, (I. def. 15)
the remainder KD is greater than the remainder GD ; (ax. 5)
that is, GD *is less than* KD ;
and because MK, DK, are drawn to the point K within the triangle MLD, from M, D, the extremities of its side MD,
therefore MK, KD *are less than* ML, LD ; (I. 21)
but MK is equal to ML ; (I. def. 15)
therefore the remainder DK *is less than the remainder* DL. (ax. 5.)
In like manner it may be shown that DL is less than DH ;
therefore DG is the least, and DK less than DL, and DL than DH.

Also, there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the line through the centre.

CONSTRUCTION

At the point M, in the straight line MD, make the angle DMB equal to the angle DMK, (I. 23) and join DB.

DEMONSTRATION

Then, in the triangles KMD, BMD,

because MK is equal to MB, and MD common to both,
the two sides KM, MD, are equal to the two BM, MD, each to each;
and the angle KMD is equal to the angle BMD; (constr.)
therefore the base DK *is equal to the base* DB ; (I. 4)
but besides DB there can be no straight line drawn from D to the circumference equal to DK.
For, if there can, let it be DN;
and because DK is assumed to be equal to DN, and DK has been shown to be equal to DB,
therefore DB *must be equal to* DN,
that is, a line nearer to the least is equal to one more remote,
which has been proved to be impossible.
If, therefore, any point, &c. Q. E. D.

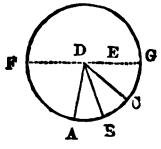
PROP. IX.—THEOREM.

If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference; then that point is the centre of the circle.

(References—III. 7.)

Let the point D be taken within the circle ABC, from which to the circumference there can be drawn more than two equal straight lines, viz. DA, DB, DC.

Then the point D shall be the centre of the circle.



CONSTRUCTION

For, if not, let E be the centre; join DE, and produce it to the circumference in FG.

DEMONSTRATION

Then FG is a diameter of the circle ABC; (I. def. 17)
and because in FG, the diameter of the circle ABC, there is taken the point D, not the centre,
therefore DG must be the greatest line from the point D *to the circumference, and* DC *greater than* DB, and DB than DA; (III. 7)

but these lines are likewise equal, (hyp.) which is impossible.

therefore E *is not the centre of the circle* ABC.

In like manner it may be demonstrated that no other point but D is the centre;

therefore D is the centre of the circle ABC.

Wherefore, if a point be taken, &c. Q. E. D.

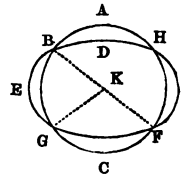
PROP. X.— THEOREM.

One circumference of a circle cannot cut another in more than two points.

(References—III. 3, 5, 9.)

Let the two circles ABC, DEF, intersect one another.

Then their circumferences cannot cut each other in more than two points.



CONSTRUCTION

If it be possible, let the circumference FAB cut the circumference DEF in more than two points, viz. in B, G, F;

take the centre K of the circle ABC, (III. 3) and join KB, KG, KF.

DEMONSTRATION

And because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG. KF,

the point K *must be the centre of the circle* DEF, (III. 9)

but K is also the centre of the circle ABC;

therefore the point K *must be the centre of two circles that cut one another,*

which is impossible. (III. 5.)

Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.

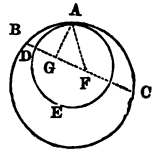
PROP. XI.—THEOREM.

If one circle touch another internally in any point, the straight line which joins their centres, being produced, shall pass through that point.

(References — Prop. I. 20.)

Let the circle ADE touch the circle ABC internally in the point A.

Then the straight line which joins their centres, being produced, shall pass through the point of contact A.



CONSTRUCTION

For, if not, let it fall otherwise, if possible, as FGDB; let F be the centre of the circle ABC, and G the centre of ADE. Join AF and AG.

DEMONSTRATION

Because two sides of a triangle are together greater than the third side,

therefore AG, GF, *are greater than* FA; (I. 20)

but FA is assumed to be equal to FB, (I. def. 15) both being from the same centre;

therefore AG, GF, *must be greater than* FB

take away the common part FG;

then the remainder AG *must be greater than the remainder* GB;

but AG is assumed to be equal to GD; (I. def. 15)

therefore GD *must be greater than* GB,

the less than the greater, which is impossible;

therefore the straight line which joins the centres, being produced, cannot fall otherwise than upon the point A.

that is, it must pass through the point of contact of the two circles.

Therefore, if two circles, &c. Q. E. D.

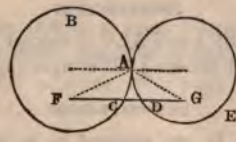
PROP. XII.— Theorem.

If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.

(References — Prop. I. 20.)

Let the two circles ABC, ADE, touch each other externally in the point A.

Then the straight line which joins their centres, shall pass through the point of contact A.



CONSTRUCTION

For, if not, let, if possible, F and G be the centres, and the line joining them pass otherwise as FCDG ; and join FA, AG.

DEMONSTRATION

Then, because F is assumed to be the centre of the circle ABC,

therefore AF *must be equal to* FC ;

also, because G is assumed to be the centre of the circle ADE

therefore AG *must be equal to* GD ;

and therefore FA, AG, must be equal to FC, DG ;

wherefore the whole FG *must be greater than* FA, AG;

but FG is also less than FA, AG ; (I. 20) which is impossible.

therefore the straight line which joins the centres of the circles, shall not pass otherwise than through A ;

that is, it must pass through the point of contact of the two circles.

Therefore, if two circles, &c. Q. E. D.

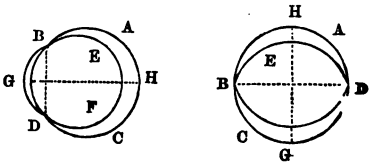
PROP. XIII.— Theorem.

One circle cannot touch another in more points than one, whether it touches it in the inside or the outside.

(*References* — Prop. I. 10, 11 ; III. 1 ; Cor. 2, 11.)

First. Let the circle EBF touch the circle ABC internally in the point B.

Then EBF cannot touch ABC in any other point.



CONSTRUCTION

If it be possible, let EBF touch ABC in another point D;
join BD, and draw GH bisecting BD at right angles. (I. 10.)

DEMONSTRATION

Because the points B, D, are assumed to be in the circumference of each of the circles,

the straight line BD *must fall within each of the circles* EBF, ABC; (III. 2)

and their centres are in the straight line GH, which bisects BD at right angles; (III. 1, Cor.)

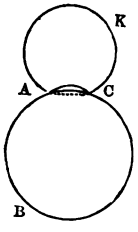
therefore GH *must pass through the point of contact;* (III. 11)

but GH does not pass through the point of contact, because the points B, D, are without the straight line GH; which is absurd;

therefore the circle EBF cannot touch the circle ABC on the inside in more points than one.

Secondly. Let the circle ACK touch the circle ABC externally in the point A.

Then ACK cannot touch ABC in any other point.



CONSTRUCTION

If it be possible, let ACK touch ABC in another point C. Join AC.

DEMONSTRATION.

ecause the points A, C, are assumed to be in the circumference
circle ACK,

aight line AC *must fall within the circle* ACK; (III. 2)

circle ACK is without the circle ABC; (hyp.)

ore the straight line AC *must be without the circle* ABC;

ause the points A, C, are assumed to be in the circumference of
rcle ABC,

aight line AC *must be within the circle;* (III. 2)

s absurd;

sfore the circle ACK cannot touch the circle ABC the outside in more than one point.

ıas been shown that one circle cannot touch another on the inside
re points than one.

ore one circle, &c. Q. E. D.

PROP. XIV.— THEOREM.

l straight lines in a circle are equally distant from the centre; and ly, those which are equally distant from the centre, are equal to ther.

(References — Prop. I. 12, 47; III. 1, 3, def. 4.)

he straight lines AB, CD, in the circle ABDC, be equal to one

a they shall be equally distant from the centre.

CONSTRUCTION

the centre of the circle ABDC, (III. 1)

m it draw EF, EG, perpendiculars to AB, CD, (I. 12) and join EC.

DEMONSTRATION

ecause the straight line EF passing through the centre, cuts the ght line AB, which does not pass through the centre, at right s, it also bisects it; (III. 3)

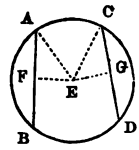
ore AF *is equal to* FB, *and* AB *double of* AF;

same reason

s double of CG;

is equal to CD; (hyp.)

ore AF is equal to CG. (AX. 7.)



And because AE is equal to EC, (I. def. 15)

the square of AE *is equal to the square of* EC;

but the squares of AF, FE, are equal to the square of AE, because the angle AFE is a right angle; (I. 47)

and for the same reason, the squares of EG, GC, are equal to the square of EC;

therefore the squares of AF, FE, *are equal to the squares of* CG, GE;

of which the square of AF is equal to the square of CG, because AF is equal to CG;

therefore the remaining square of EF *is equal to the remaining square of* EG, (ax. 3)

and therefore the straight line EF *is equal to* EG;

but 'straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal;' (III. def. 4)

therefore AB, CD, are equally distant from the centre E.

Next, let the straight lines AB, CD, be equally distant from the centre, that is, let FE be equal to EG.

Then AB shall be equal to CD.

The same construction being made, it may be demonstrated as before,

that AB *is double of* AF, *and* CD *double of* CG,

and that the squares of EF, FA, *are equal to the squares of* EG, GC;

of which the square of FE is equal to the square of EG, because FE is equal to EG; (hyp.)

therefore the remaining square of AF *is equal to the remaining square of* CG; (ax. 3)

and therefore the straight line AF *is equal to* CG;

but AB was shown to be double of AF, and CB double of CG;

wherefore AB is equal to CD.

Therefore, equal straight lines, &c. Q. E. D.

PROP. XV.—Theorem.

The diameter is the greatest straight line in a circle, and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

(References—Prop. I. 12, 20; III. def. 5.)

Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG.

Then AD shall be greater than any straight line BC, which is not a diameter,
and BC shall be greater than FG.

CONSTRUCTION

From the centre E draw EH, EK, perpendiculars to BC, FG, (I. 12) and join EB, EC, EF.

DEMONSTRATION

Then, because AE is equal to EB, and ED to EC, (I. def. 15)
therefore AD *is equal to* EB, EC; (ax. 2)
but EB, EC, are greater than BC; (I. 20)
wherefore also AD is greater than BC.

And because BC is nearer to the centre than FG, (hyp.)
therefore EH *is less than* EK; (III. def. 5)
but, as was demonstrated in the preceding proposition,
BC *is double of* BH, *and* FG *double of* FK,
and the squares of EH, HB, *are equal to the squares of* EK, KF,
but the square of EH is less than the square of EK, because EH is less than EK;
therefore the square of BH *is greater than the square of* FK,
and the straight line BH *greater than* FK;
and therefore BC is greater than FG.

Next, let BC be greater than FG.

Then BC shall be nearer to the centre than FG, that is, EH shall be less than EK.

The same construction being made, because BC is greater than FG,

therefore BH is greater than KF;

and the squares of BH, HE, *are equal to the squares of* FK, KE,

but the square of BH is greater than the square of FK, because BH is greater than FK;

therefore the square of EH *is less than the square of* EK,
and the straight line EH *less than* EK.

and therefore BC is nearer to the centre than FG. (III. def. 5.)

Wherefore, the diameter, &c. Q. E. D.

PROP. XVI.—THEOREM.

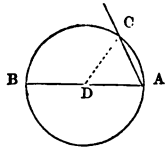
The straight line which is drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle;

and no straight line can be drawn from the extremity between that straight line and the circumference, which does not cut the circle, or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.

(References—Prop. I. 5, 12, 17, 19; III. 2, def. 2.)

Let ABC be a circle, the centre of which is D, and the diameter AB.

Then the straight line drawn at right angles to AB from its extremity A, shall fall without the circle.



CONSTRUCTION

For, if it does not, let it fall, if possible, *within* the circle, as AC; *and draw DC* to the point C, where it meets the circumference.

DEMONSTRATION

Because DA is equal to DC, (I. def. 15)

the angle DAC *is equal to the angle* ACD; (I. 5)

but DAC is a right angle, (hyp.)

therefore ACD must be a right angle,

and therefore the angles DAC, ACD, *must be equal to two right angles;*

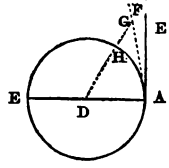
which is impossible; (I. 17)

therefore the straight line drawn from A *at right angles to* BA, *does not fall within the circle.*

In the same manner it may be demonstrated that it does not fall *upon* the circumference;

therefore the straight line from A at right angles to BA, must fall without the circle, as AE.

Also, between the straight line AE and the circumference, no straight line can be drawn from the point A which does not cut the circle.



CONSTRUCTION

For, if possible, let FA be between them,

and from the point D draw DG perpendicular to FA, (I. 12)

and let it meet the circumference in H.

DEMONSTRATION

Because AGD is assumed to be a right angle, and DAG less than a right angle; (I. 17)

therefore DA *must be greater than* DG; (I. 19)

but DA is equal to DH; (I. def. 15)

therefore DH *must be greater than* DG,

the less than the greater, which is impossible;

therefore no straight line can be drawn from the point A, between AE and the circumference, which does not cut the circle;

or, which amounts to the same thing, however great an acute angle a *straight line makes* with the diameter at the point A, or however

small an angle it makes with AE, the circumference shall pass between that straight line and the perpendicular AE.

'And this is all that is to be understood, when in the Greek text, and in translations from it, the angle of the semicircle is said to be greater than any acute rectilineal angle, and the remaining angle less than any rectilineal angle.' Q. E. D.

Cor.—From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle, from the extremity of it, touches the circle; (III. def. 2) and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it. (III. 2.)

'Also it is evident that there can be but one straight line which touches the circle in the same point.'

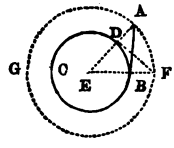
PROP. XVII.—PROBLEM.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

(References—Prop. I. 4, 11; III. 1, 16, cor.)

First, let A be a given point without the given circle BCD.

It is required to draw a straight line from A which shall touch the circle.



CONSTRUCTION

Find the centre E of the circle, (III. 1) and join AE;
and from the centre E, at the distance AE, describe the circle AFG;
from the point D draw FD at right angles to AE; (I. 11) and join EBF, AB.

Then AB shall touch the circle BCD.

DEMONSTRATION

Because E is the centre of the circles BCD, AFG;

therefore EA *is equal to* EF, *and* ED *to* EB; (I. def. 15)

hence, in the two triangles AEB, FED,

ecause the two sides AE, EB, are equal to the two FE, ED, each to each;

nd they contain the angle at E common to both triangles;

therefore, the base DF is equal to the base AB,

and the triangle FED *to the triangle* AEB,

and the other angles to the other angles;

therefore the angle EBA *is equal to the angle* EDF; (I. 4)

ut EDF is a right angle, (constr.)

wherefore EBA *is a right angle*; (ax. 1)

nd EB is drawn from the centre;

ut 'a straight line drawn from the extremity of a diameter, at right angles to it, touches the circle;' (III. 16, cor.)

therefore AB touches the circle,

nd it is drawn from the given point A.

ut if the given point be in the circumference of the circle, as the point D,

raw DE to the centre E, and DF at right angles to DE;

then DF touches the circle. (III. 16, cor.) Q. E. F.

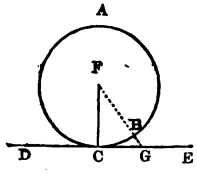
PROP. XVIII.—THEOREM.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

(References—Prop. I. 17, 19; III. 1.)

Let the straight line DE touch the circle ABC in the point C; take the centre F, (III. 1) and draw the straight line FC.

Then FC shall be perpendicular to DE.



CONSTRUCTION

For, if it be not, from the point F draw FBG perpendicular to DE.

DEMONSTRATION

And because FGC is assumed to be a right angle,

therefore GCF *must be an acute angle*; (I. 17)

and to the greater angle the greater side is opposite; (I. 19)

therefore FC *must be greater than* FG,

but FC is equal to FB, (I. def. 15)

therefore FB *must be greater than* FG,

the less than the greater, which is impossible;

wherefore FG *is not perpendicular to* DE.

In the same manner it may be shown, that no other is perpendicular to it besides FC;

that is, FC is perpendicular to DE.

Therefore, if a straight line, &c. Q. E. D.

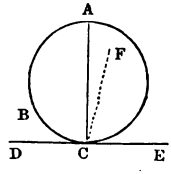
PROP. XIX.—THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

(References — Prop. III. 18.)

Let the straight line DE touch the circle ABC in C; and from C let CA be drawn at right angles to DE.

Then the centre of the circle shall be in CA.



CONSTRUCTION

For, if not, let F be the centre, if possible, and join CF.

DEMONSTRATION

Because DE touches the circle ABC, and FC is drawn from the assumed centre to the point of contact,

therefore FC must be perpendicular to DE; (III. 18)

and therefore FCE *must be a* right angle;

but ACE is a right angle, (hyp.)

therefore the angle FCE *must be equal to the angle* ACE, (ax. 1)

the less to the greater, which is impossible.

wherefore F *is not the centre of the circle* ABC.

In the same manner, it may be shown that no other point which is not in CA is the centre;

that is, the centre of the circle is in CA.

Therefore, if a straight line, &c. Q. E. D.

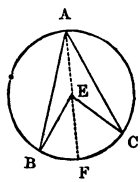
PROP. XX.—THEOREM.

The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

(References—Prop. I. 5, 32.)

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base.

Then the angle BEC shall be double of the angle BAC.



First, let the centre E of the circle be *within* the angle BAC.

CONSTRUCTION

Join AE, and produce it to the circumference in F.

DEMONSTRATION

Because EA is equal to EB,

the angle EAB is equal to the angle EBA, (I. 5)

therefore the angles EAB, EBA *are double of the angle* EAB;

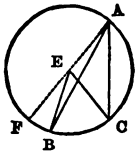
but the angle BEF is equal to the angles EAB, EBA; (I. 32)

therefore also the angle BEF *is double of the angle* EAB;

for the same reason, the angle FEC is double of the angle EAC;

therefore the whole angle BEC is double of the whole angle BAC.

Secondly, let the centre E of the circle be *without* the angle BAC.



CONSTRUCTION

Join AE, and produce it to meet the circumference in F.

DEMONSTRATION

It may be demonstrated, as in the first case,

that the angle FEC *is double of the angle* FAC,
and that FEB, *a part of* FEC, *is double of* FAD, *a part of* FAC;

therefore the remaining angle BEC is double of the remaining angle BAC.

Therefore, the angle at the centre, &c. Q. E. D.

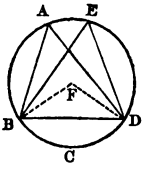
PROP. XXI.—THEOREM.

The angles in the same segment of a circle are equal to one another.

(References—Prop. III. 1, 20.)

Let ABCD be a circle, and BAD, BED, angles in the same segment BAED.

Then the angles BAD, BED, shall be equal to one another.



First, let the segment BAED be greater than a semicircle.

CONSTRUCTION

Take F the centre of the circle ABCD, (III. 1) and join BF, FD.

DEMONSTRATION

And, because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same part of the circumference, viz. BCD for their base;

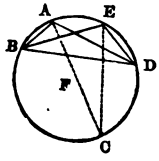
therefore the angle BFD *is double of the angle* BAD; (III. 20)

for the same reason,

the angle BFD *is double of the angle* BED;

therefore the angle BAD is equal to the angle BED. (ax. 7.)

Next, let the segment BAED be not greater than a semicircle.



CONSTRUCTION

Draw AF to the centre, and produce it to C, and join CE.

DEMONSTRATION

Then the segment BADC is greater than a semicircle,

therefore by the first case, the angles BAC, BEC, *in it, are equal;*

for the same reason, because CBED is greater than a semicircle,

the angles CAD, CED, *are equal;*

therefore the whole angle BAD is equal to the whole angle BED. (ax. 2.)

Wherefore, the angles in the same segment, &c. Q. E. D.

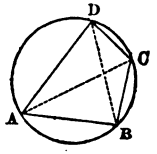
PROP. XXII.—THEOREM.

The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.

(References—Prop. I. 32; III. 21.)

Let ABCD be a quadrilateral figure in the circle ABCD.

Then any two of its opposite angles shall be together equal to two right angles.



CONSTRUCTION

Draw the diagonals AC and DB.

DEMONSTRATION

Because the three angles of every triangle are equal to two right angles, (I. 32)

the three angles of the triangle CAB, *viz. the angles* CAB, ABC, BCA, *are equal to two right angles;*

but the angle CAB is equal to the angle CDB, (III. 21) because they are in the same segment CDAB;

and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB;

therefore the whole angle ADC *is equal to the angles* CAB, ACB;

to each of these equals add the angle ABC;

therefore the three angles ABC, CAB, BCA, *are equal to the two angles* ABC, ADC; (ax. 2)

but ABC, CAB, BCA, are equal to two right angles;

therefore also the angles ABC, ADC, are equal to two right angles.

In the same manner, it may be shown that

the angles BAD, DCB, are equal to two right angles.

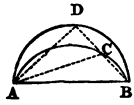
Therefore, the opposite angles, &c. Q. E. D.

PROP. XXIII.—THEOREM.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

(References — Prop. I. 16; III. 10, def. 11.)

If it be possible, let the two similar segments of circles, viz. ACB, ADB, be upon the same side of the same straight line AB, not coinciding with one another.



CONSTRUCTION

Then, because the circle ACB cuts the circle ADB, in the two points A, B,

they cannot cut one another in any other point; (III. 10)

therefore one of the segments ACB, ADB, *must fall within the other.*

Let ACB fall within ADB, and draw the straight line BCD, and join CA, DA.

DEMONSTRATION

Because the segment ACB is assumed to be similar to the segment ADB, (hyp.)

and that similar segments of circles contain equal angles; (III. def. 11)

the angle ACB *must be equal to the angle* ADB,

the exterior to the interior, which is impossible; (I. 16)

Therefore there cannot be two similar segments of circles upon the same side of the same line, which do not coincide. Q. E. D.

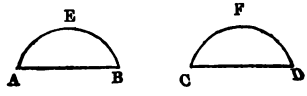
PROP. XXIV.— THEOREM.

Similar segments of circles upon equal straight lines are equal to one another.

(References — Prop. III. 23.)

Let AEB, CFD, be similar segments of circles upon the equal straight lines AB, CD.

Then the segment AEB shall be equal to the segment CFD.



DEMONSTRATION

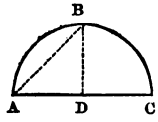
For if the segment AEB be applied to the segment CFD, so that the point A may be on C, and the straight line AB on CD;
then, because AB is equal to CD,
the point B *shall coincide with the point* D;
and the straight line AB coinciding with CD,
the segment AEB *coincides with the segment* CFD, (III. 23)
and therefore the segment AEB is equal to the segment CFD. (ax. 8.)
Wherefore, similar segments, &c. Q. E. D.

PROP. XXV.—PROBLEM.

A segment of a circle being given, to describe the circle of which it is the segment.

(References — Prop. I. 4, 6, 10, 11, 23; III. 9.)

Let ABC be the given segment of a circle.
It is required to describe the circle of which it is the segment.



CONSTRUCTION

Bisect AC in D, (I. 10) and from the point D draw DB at right angles to AC; (I. 11) and join AB.

First, let the angles ABD, BAD, be equal to one another.
Then D shall be the centre of the circle required.

DEMONSTRATION

Because the angle ABD is equal to the angle BAD, (hyp.)
therefore DA *is equal to* DB; (I. 6)
but DA is equal to DC, (constr.)
therefore DB is equal to DC,
and the three straight lines DA, DB, DC, are all equal;

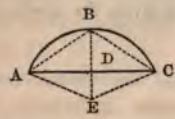
and because from the point D more than two equal straight lines DA, DB, DC, can be drawn to the circumference ABC,

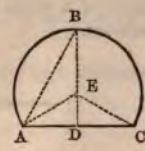
therefore D is the centre of the circle. (III. 9.)

Hence, if with the centre D, at the distance of any of the three, DA, DB, DC, a circle be described,

it will pass through the other two points, and be the circle required.

Secondly, let the angles ABD, BAD, be not equal to one another.





CONSTRUCTION

At the point A, in the straight line BA, make the angle BAE equal to the angle ABD, (I. 23)

produce BD, if necessary, to E, and join EC.

Then E shall be the centre of the circle required.

DEMONSTRATION

Because the angle ABE is equal to the angle BAE, (constr.)

the straight line BE *is equal to* EA ; (I. 6)

and in the triangles ADE, CDE,

because AD is equal to DC, and DE common to both,

the two sides AD, DE, are equal to the two CD, DE, each to each ;

and the angle ADE is equal to the angle CDE, for each of them is a right angle ; (constr.)

therefore the base AE *is equal to the base* EC ; (I. 4)

but AE was shown to be equal to EB,

wherefore also EB is equal to EC ; (ax. 1)

and the three straight lines EA, EB, EC, *are therefore equal to one another ;*

and because from the point E, more than two equal straight lines, EA, EB, EC, can be drawn to the circumference ABC,

therefore E is the centre of the circle. (III. 9.)

Hence, if with the centre E, at the distance of any of the three AE, EB, EC, a circle be described,

it will pass through the other point, and be the circle required.

In the *first* case, it is evident that, because the centre D is in AC,

therefore the segment ABC *is a semicircle.*

In the *second* case, if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC,

which is therefore less than a semicircle;

but if the angle ABD be less than BAD, the centre E falls within the segment ABC,

which is therefore greater than a semicircle.

Wherefore a segment of a circle being given, the circle is described of which it is a segment. Q. E. F.

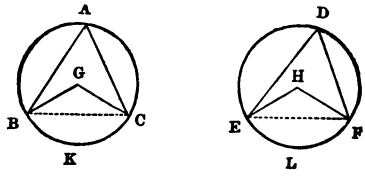
PROP. XXVI.—THEOREM.

In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or at the circumferences.

(References—Prop. I. 4; III. 24, def. 1, 11.)

Let ABC, DEF, be equal circles, having the equal angles BGC, EHF, at their centres, and BAC, EDF, at their circumferences.

Then the circumference BKC shall be equal to the circumference ELF.



CONSTRUCTION

Join BC and EF.

DEMONSTRATION

Because the circles ABC, DEF are equal, the straight lines drawn from their centres are equal; (III. def. 1)

therefore the two sides BG, GC, *are equal to the two* EH, HF, *each to each;*

and the angle at G is equal to the angle at H; (hyp.)

therefore the base BC *is equal to the base* EF. (I. 4.)

And because the angle at A is equal to the angle at D, (hyp.)

the segment BAC *is similar to the segment* EDF; (III. def. 11)

and they are upon equal straight lines BC, EF;

but similar segments of circles upon equal straight lines are equal to one another; (III. 24)

therefore the segment BAC *is equal to the segment* EDF;

but the whole circle ABC is equal to the whole DEF; (hyp.)

therefore the remaining segment BKC *is equal to the remaining segment* ELF, (ax. 3)

and the circumference BKC to the circumference ELF.

Wherefore, in equal circles, &c. Q. E. D.

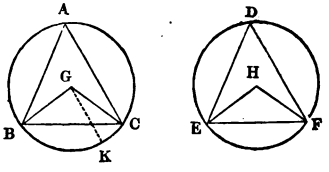
PROP. XXVII.—THEOREM.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or at the circumferences.

(References—Prop. I. 23; III. 20, 26.)

Let the angles BGC, EHF, at the centres, and BAC, EDF, at the circumferences of the equal circles ABC, DEF, stand upon equal circumferences BC, EF.

Then the angle BGC shall be equal to the angle EHF, and the angle BAC to the angle EDF.



If the angle BGC be equal to the angle EHF,

it is manifest that the angle BAC *is also equal to* EDF. (III. 20, and ax. 7.)

But, if not, one of them is greater than the other; let BGC be the greater.

CONSTRUCTION

At the point G, in the straight line BG, make the angle BGK equal to *the angle EHF.* (I. 23.)

DEMONSTRATION

Because equal angles stand upon equal circumferences when they are at the centre; (III. 26)

therefore the circumference BK *must be equal to the circumference* EF;

but EF is equal to BC; (hyp.)

therefore also BK *must be equal to* BC, (ax. 1)

the less to the greater, which is impossible;

therefore the angle BGC *is not unequal to the angle* EHF;

that is, the angle BGC is equal to the angle EHF.

And the angle at A is half of the angle BGC, (III. 20)

and the angle at D is half of the angle EHF;

therefore the angle at A is equal to the angle at D. (ax. 7.)

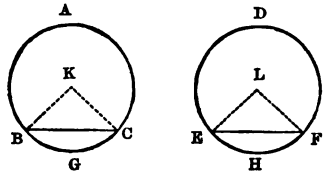
PROP. XXVIII.—THEOREM.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

(References—Prop. I. 8; III. 1, 26, def. 1.)

Let ABC, DEF, be equal circles, and BC, EF, equal straight lines in them, which cut off the two greater circumferences BAC, EDF, and the two less, BGC, EHF.

Then the greater circumference BAC shall be equal to the greater EDF, and the less BGC to the less EHF.



CONSTRUCTION

Take K, L, the centres of the circles, (III. 1) and join BK, KC, EL, LF.

DEMONSTRATION

Because the circles ABC, DEF, are equal, the straight lines from their *centres* are equal; (III. def. 1)

therefore BK, KC, *are equal to* EL, LF, *each to each*;

and the base BC is equal to the base EF; (hyp.)

therefore the angle BKC *is equal to the angle* ELF; (I. 8)

but equal angles stand upon equal circumferences, when they are at the centres; (III. 26)

therefore the circumference BGC is equal to the circumference EHF;

but the whole circle ABC is equal to the whole EDF; (hyp.)

therefore the remaining parts of the circumferences are equal,

that is, BAC is equal to EDF.

Therefore, in equal circles, &c. Q. E. D.

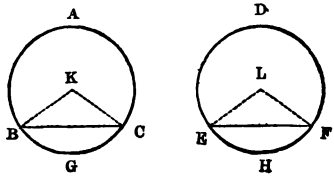
PROP. XXIX.—THEOREM.

In equal circles equal circumferences are subtended by equal straight lines.

(References—Prop. I. 4; III. 1, 27.)

Let ABC, DEF, be equal circles, and let the circumferences BGC, EHF, also be equal; and join BC, EF.

Then the straight line BC shall be equal to the straight line EF.



CONSTRUCTION

Take K, L, the centres of the circles, (III. 1) and join BK, KC, EL, LF.

DEMONSTRATION

Because the circumference BGC is equal to the circumference EHF,

therefore the angle BKC *is equal to the angle* ELF; (III. 27)

and because the circles ABC, DEF, are equal, the straight lines from *their centres* are equal; (III. def. 1)

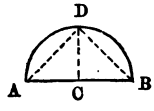
therefore BK, KC, *are equal to* EL, LF, *each to each;*
and they contain equal angles;
therefore the base BC is equal to the base EF. (I. 4.)
Therefore, in equal circles, &c. Q. E. D.

PROP. XXX.—PROBLEM.

To bisect a given circumference, that is, to divide it into two equal parts.

(References—Prop. I. 4, 10, 11; III. 1 cor., 28.)

Let ADB be the given circumference.
It is required to bisect it.



CONSTRUCTION

Join AB, and bisect it in C; (I. 10)
from the point C draw CD at right angles to AB. (I. 11.)
Then the circumference ADB shall be bisected in the point D.
Join AD, DB.

DEMONSTRATION

In the triangles ACD, BCD,
because AC is equal to CB, and CD common to both,
the two sides AC, CD, are equal to the two BC, CD, each to each;
and the angle ACD is equal to the angle BCD, because each of them is a right angle;
therefore the base AD *is equal to the base* BD; (I. 4)
but equal straight lines cut off equal circumferences, (III. 28) the greater equal to the greater, and the less to the less;
and AD, DB, are each of them less than a semicircle, because DC passes through the centre; (III. 1 cor.)
therefore the circumference AD *is equal to the circumference* DB;
wherefore the circumference ADB is bisected in D.
Q. E. F.

PROP. XXXI.—THEOREM.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

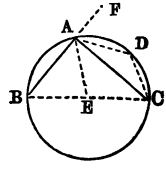
(References—Prop. I. 5, 17, 32; III. 22.)

Let ABCD be a circle, of which the diameter is BC, and the centre E; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC.

Then the angle in the semicircle BAC shall be a right angle;

and the angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle;

and the angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.



CONSTRUCTION

Join AE, and produce BA to F.

DEMONSTRATION

Because BE is equal to EA, (I. def. 15)

the angle EAB *is equal to* EBA; (I. 5)

also, because AE is equal to EC,

the angle EAC is equal to the angle ECA;

wherefore the whole angle BAC *is equal to the two angles* ABC, ACB;

but FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB; (I. 32)

therefore the angle BAC *is equal to the angle* FAC; (ax. 1)

and each of them is therefore a right angle; (I. def. 10)

wherefore the angle BAC in a semicircle is a right angle.

And because the two angles ABC, BAC, of the triangle ABC, are *together less than* two right angles, (I. 17)

and that BAC has been proved to be a right angle;

therefore ABC *is less than a right angle;*

and therefore the angle in a segment ABC, greater than a semicircle, is less than a right angle.

And because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal to two right angles; (III. 22)

therefore the angles ABC, ADC, *are equal to two right angles;*

and ABC has been proved to be less than a right angle;

wherefore the other ADC is greater than a right angle.

Besides, it is manifest that the circumference of the greater segment ABC falls without the right angle CAB, but the circumference of the less segment ADC falls within the right angle CAF.

'And this is all that is meant, when in the Greek text, and the translations from it, the angle of the greater segment is said to be greater, and the angle of the less segment is said to be less than a right angle.'

Cor.—From this it is manifest that if one angle of a triangle be equal to the other two it is a right angle, because the angle adjacent to it is equal to the same two; (I. 32) and when the adjacent angles are equal, they are right angles. (I. def. 10.)

PROP. XXXII.—THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle;

then the angles made by this line with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

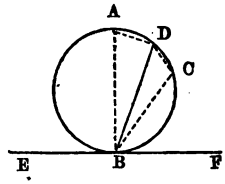
(References — Prop. I. 11, 13, 32; III. 19, 22, 31.)

Let the straight line EF touch the circle ABCD in B, and from the point B let the straight line BD be drawn cutting the circle.

Then the angles which BD makes with the touching line EF shall be equal to the angles in the alternate segments of the circle;

that is, the angle FBD shall be equal to the angle which is in the segment DAB,

and the angle DBE to the angle in the segment BCD.



CONSTRUCTION

From the point B draw BA at right angles to EF; (I. 11)
and take any point C in the circumference BD, and join AD, DC, CB.

DEMONSTRATION

Because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B,

the centre of the circle ABCD is in BA; (III. 19)

therefore the angle ADB *in a semicircle, is a right angle,* (III. 31)

and consequently the other two angles BAD, ABD, *are equal to a right angle;* (I. 32)

but ABF is likewise a right angle; (constr.)

therefore the angle ABF *is equal to the angles* BAD, ABD; (ax. 1)

take from these equals the common angle ABD;

therefore the remaining angle DBF is equal to the angle BAD, which is in the alternate segment of the circle. (ax. 3.)

And because ABCD is a quadrilateral figure in a circle,

the opposite angles BAD, BCD, *are equal to two right angles;* (III. 22)

but the angles DBF, DBE, are likewise equal to two right angles; (I. 13)

therefore the angles DBF, DBE, *are equal to the angles* BAD, BCD;

and DBF has been proved equal to BAD;

therefore the remaining angle DBE is equal to the angle BCD, which is in the alternate segment of the circle. (ax. 3.)

Wherefore, if a straight line, &c. Q. E. D.

PROP. XXXIII.—PROBLEM.

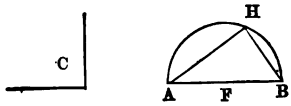
Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

(References—Prop. I. 4, 10, 11, 23, 31; III. 16 cor., 31, 32.)

Let AB be the given straight line, and the angle C the given rectilineal angle.

It is required to describe upon the given straight line AB, a segment of a circle, containing an angle equal to the angle C.

First, let the angle C be a right angle.



CONSTRUCTION

Bisect AB in F, (I. 10) and from the centre F, at the distance FB, describe the semicircle AHB.

Then AHB shall be the segment required.

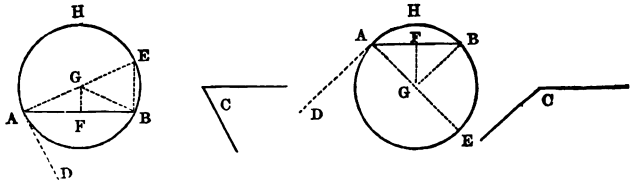
DEMONSTRATION

Because AHB is a semicircle,

therefore the angle AHB *in it is a right angle, and therefore equal to* C; (III. 31)

wherefore the angle in the segment AHB is equal to the given angle C.

Secondly, let C be not a right angle.



CONSTRUCTION

At the point A in the straight line AB, make the angle BAD equal to the angle C, (I. 23)

and from the point A draw AE at right angles to AD; (I. 11)

bisect AB in F, (I. 10) and from F draw FG at right angles to AB, *(I. 11) and* join GB.

Then AHB is the segment required.

DEMONSTRATION

In the triangles AFG, BFG,
because AF is equal to FB, and FG common to both,
the two sides AF, FG, are equal to the two BF, FG, each to each;
and the angle AFG is equal to the angle BFG; (I. def. 10)

therefore the base AG *is equal to the base* GB; (I. 4)

and the circle described from the centre G, *at the distance* GA, *shall pass through the point* B;

let this be the circle AHB;

and because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE,

therefore AD *touches the circle* AHB; (III. 16, cor.)

and because AB drawn from the point of contact A cuts the circle,

the angle DAB *is equal to the angle in the alternate segment* AHB; (III. 32)

but the angle DAB is equal to the angle C; (constr.)

therefore the angle C is equal to the angle in the segment AHB.

Wherefore, upon the given straight line AB, the segment AHB of a circle is described which contains an angle equal to the given angle C. Q. E. F.

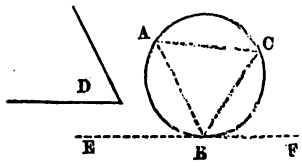
PROP. XXXIV.—PROBLEM.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

(References — Prop. I. 23; III. 17, 32.)

Let ABC be the given circle, and D the given rectilineal angle.

It is required to cut off a segment from the circle ABC that shall contain an angle equal to the angle D.



CONSTRUCTION

Draw the straight line EF touching the circle ABC in the point B, *(III. 17)*

and at the point B, in the straight line BF make the angle FBC equal to the angle D. (I. 23.)

Then the segment BAC shall contain an angle equal to the given angle D.

DEMONSTRATION

Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact, B,

therefore the angle FBC *is equal to the angle in the alternate segment* BAC *of the circle*; (III. 32)

but the angle FBC is equal to the angle D; (constr.)

therefore the angle in the segment BAC is equal to the angle D. (AX. 1.)

Wherefore the segment BAC is cut off from the given circle ABC containing an angle equal to the given angle D. Q. E. F.

PROP. XXXV.—THEOREM.

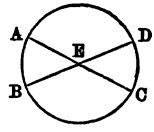
If two straight lines within a circle cut one another;

then the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

(References — Prop. I. 12, 47; II. 5; III. 1, 3.)

Let the two straight lines AC, BD, within the circle ABCD, cut one another in the point E.

Then the rectangle contained by AE, EC, shall be equal to the rectangle contained by BE, ED.



First, let AC, BD, pass each of them through the centre.

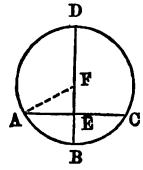
DEMONSTRATION

Then because E is the centre,

therefore AE, EC, BE, ED, *are all equal to one another*, (I. def. 15)

and therefore the rectangle AE, EC, is equal to the rectangle BE, ED.

Secondly, let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre, at right angles in the point E.



CONSTRUCTION

Then if BD be bisected in F, F is the centre of the circle ABCD. Join AF.

DEMONSTRATION

Because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E,

therefore AE, EC, *are equal to one another*, (III. 3)

and because the straight line BD is cut into two equal parts in the point F, and into unequal parts in the point E,

the rectangle BE, ED, *together with the square of* EF, *is equal to the square of* FB; (II. 5) *that is to the square of* FA;

but the squares of AE, EF, are equal to the square of FA; (I. 47)

therefore the rectangle BE, ED, *together with the square of* EF, *is equal to the squares of* AE, EF; (ax. 1)

take away the common square of EF,

and the remainders are equal, that is,

the rectangle BE, ED, is equal to the square of AE, that is to the rectangle AE, EC.

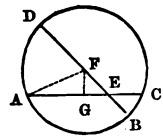
Thirdly, let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles.

CONSTRUCTION

Then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and from F draw FG perpendicular to AC. (I. 12.)

DEMONSTRATION

Then *AG is equal to* GC; (III. 3)



therefore the rectangle AE, EC, *together with the square of* EG, *is equal to the square of* AG; (II. 5)

to each of these equals add the square of GF;

therefore the rectangle AE, EC, *together with the squares of* EG, GF, *is equal to the squares of* AG, GF; (ax. 2)

but the squares of EG, GF, are equal to the square of EF; (I. 47)

and the squares of AG, GF, are equal to the square of AF;

therefore the rectangle AE, EC, *together with the square of* EF, *is equal to the square of* AF; *that is, to the square of* FB;

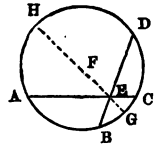
but the square of FB is equal to the rectangle BE, ED, together with the square of EF; (II. 5)

therefore the rectangle AE, EC, *together with the square of* EF, *is equal to the rectangle* BE, ED, *together with the square of* EF; (ax. 1)

take away the common square of EF,

and therefore the remaining rectangle AE, EC, is equal to the remaining rectangle BE, ED. (ax. 3.)

Lastly, let neither of the straight lines AC, BD, pass through the centre.



CONSTRUCTION

Take the centre F, (III. 1) and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH.

DEMONSTRATION

Because the rectangle AE, EC, is equal, as has been shown, to the rectangle GE, EH;

and for the same reason, the rectangle BE, ED, is equal to the same *rectangle GE*, EH;

therefore the rectangle AE, BC, is equal to the rectangle BE, ED. (ax. 1.)

Wherefore, if two straight lines, &c. Q. E. D.

PROP. XXXVI.— THEOREM.

If from any point without a circle, two straight lines be drawn, one of which cuts the circle, and the other touches it;

then the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

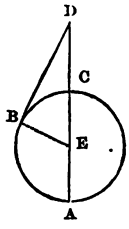
(References — Prop. I. 12, 47; II. 6; III. 1, 3, 18.)

Let D be any point without the circle ABC; and let DCA, DB, be two straight lines drawn from it, of which DCA cuts the circle, and DB touches the same.

Then the rectangle AD, DC, shall be equal to the square of BD.

Either DCA passes through the centre, or it does not.

First, let it pass through the centre E.



CONSTRUCTION

From the point E to B, draw the straight line EB.

DEMONSTRATION

Then the angle EBD is a right angle; (III. 18)

and because the straight line AC is bisected in E, and produced to the point D,

the rectangle AD, DC, *together with the square of* EC, *is equal to the square of* ED; (II. 6)

but CE is equal to EB;

therefore the rectangle AD, DC, *together with the square of* EB, *is equal to the square of* ED;

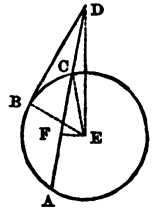
but the square of ED is equal to the squares of EB, BD, because EBD is a right angle; (I. 47)

therefore the rectangle AD, DC, *together with the square of* EB, *is equal to the squares of* EB, BD; (ax. 1)

take away the common square of EB;

therefore the remaining rectangle AD, DC, is equal to the square of the tangent DB. (ax. 3.)

Secondly, let DCA not pass through the centre of the circle ABC.



CONSTRUCTION

Take the centre E, (III. 1) and draw EF perpendicular to AC, (I. 12) and join EB, EC, ED.

DEMONSTRATION

Because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it shall likewise bisect it; (III. 3)

therefore AF *is equal to* FC;

and because the straight line AC is bisected in F, and produced to D,

the rectangle AD, DC, *together with the square of* FC, *is equal to the square of* FD; (II. 6)

to each of these equals add the square of FE;

therefore the rectangle AD, DC, *together with the squares of* CF, FE, *is equal to the squares of* DF, FE; (ax. 2)

but the square of ED is equal to the squares of DF, FE, because EFD is a right angle; (I. 47)

and for the same reason the square of EC is equal to the squares of CF, FE;

therefore the rectangle AD, DC, *together with the square of* EC, *is equal to the square of* ED; (ax. 1)

but CE is equal to EB;

therefore the rectangle AD, DC, *together with the square of* EB, *is equal to the square of* ED,

but the squares of EB, BD, are equal to the square of ED, because EBD is a right angle; (I. 47)

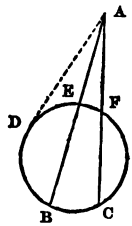
therefore the rectangle AD, DC, *together with the square of* EB, *is equal to the squares of* EB, BD;

take away the common square of EB;

therefore the remaining rectangle AD, DC, is equal to the square of DB. (AX. 3.)

Wherefore, if from any point, &c. Q. E. D.

Cor. — If from any point without a circle, there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another.



Thus the rectangle BA, AE, is equal to the rectangle CA, AF, for each of them is equal to the square of the straight line AD, which touches the circle.

PROP. XXXVII.—THEOREM.

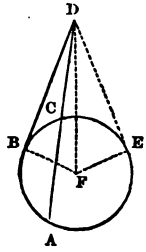
If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the square of the line which meets it;

then that line shall touch the circle.

(References — Prop. I. 8; III. 1, 16 cor., 17, 18, 36.)

Let any point D be taken without the circle ABC, and from it let two straight lines, DCA and DB, be drawn, of which DCA cuts the circle, and DB meets it.

If the rectangle AD, DC, be equal to the square of DB,
Then DB shall touch the circle.



CONSTRUCTION

Draw the straight line DE, touching the circle ABC in E; (III. 17)
find F, the centre of the circle, (III. 1) and join FE, FB, FD.

DEMONSTRATION

Then FED is a right angle; (III. 18)
and because DE touches the circle ABC, and DCA cuts it;
the rectangle AD, DC, *is equal to the square of* DE; (III. 36)
but the rectangle AD, DC, is, by hypothesis, equal to the square of DB;
therefore the square of DE is equal to the square of DB; (AX. 1)
and the straight line DE *equal to the straight line* DB;
and FE is equal to FB. (I. def. 15.)

Hence in the two triangles DEF, DBF,
because DE is equal to DB, and EF to FB,
the two sides DE, EF, are equal to the two DB, BF, each to each;
and the base FD is common to the two triangles;
therefore the angle DEF *is equal to the angle* DBF; (I. 8)
but DEF was shown to be a right angle,
therefore also DBF *is a right angle;* (AX. 1)
and FB, if produced, is a diameter,
and the 'straight line which is drawn at right angles to a diameter, from the extremity of it, touches the circle;' (III. 16 cor.)
therefore DB touches the circle ABC.

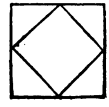
Wherefore, if from a point, &c. Q. E. D.

BOOK IV.

DEFINITIONS.

I.

A RECTILINEAL figure is said to be inscribed in another rectilineal figure, when all the angular points of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

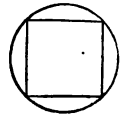


II.

In like manner a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

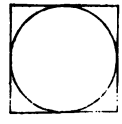
III.

A rectilineal figure is said to be inscribed in a circle, when all the angular points of the inscribed figure are upon the circumference of the circle.



IV.

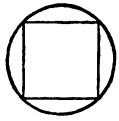
A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.



V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.



A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.

VII.

A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

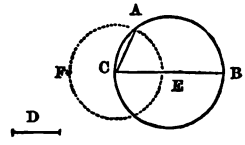
PROP. I.—PROBLEM.

In a given circle to place a straight line equal to a given straight line which is not greater than the diameter of the circle.

(References — Prop. I. 3.)

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle.

It is required to place in the circle ABC a straight line equal to D.



CONSTRUCTION

Draw BC the diameter of the circle ABC;
then, if BC is equal to D, the thing required is done;
for in the circle ABC a straight line BC is placed equal to D.
But, if it is not, BC is greater than D; (hyp.)
make CE equal to D, (I. 3)
and from the centre C, at the distance CE, describe the circle AEF and join CA.

Then CA shall be equal to D.

DEMONSTRATION

Because C is the centre of the circle AEF,
therefore CA is equal to CE; (I. def. 15)
but D is equal to CE; (constr.)

therefore D is equal to CA; (ax. 1)

Wherefore, in the circle ABC, a straight line CA is placed equal to the given straight line D, which is not greater than the diameter of the circle. Q. E. F.

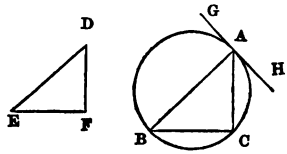
PROP. II.—Problem.

In a given circle to inscribe a triangle equiangular to a given triangle.

(References — Prop. I. 23, 32; III. 17, 32.)

Let ABC be the given circle, and DEF the given triangle.

It is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.



CONSTRUCTION

Draw the straight line GAH touching the circle in the point A; (III. 17)

and at the point A, in the straight line AH, make the angle HAC equal to the angle DEF; (I. 23)

and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC.

Then ABC shall be the triangle required.

DEMONSTRATION

Because HAG touches the circle ABC, and AC is drawn from the point of contact,

the angle HAC *is equal to the angle* ABC *in the alternate segment of the circle;* (III. 32)

but HAC is equal to the angle DEF; (constr.)

therefore also the angle ABC *is equal to* DEF; (ax. 1)

for the same reason, the angle ACB is equal to the angle DFE;

therefore the remaining angle BAC *is equal to the remaining angle* EDF; (I. 32, and ax. 1)

wherefore the triangle ABC is equiangular to the triangle DEF,

and it is inscribed in the circle ABC. Q. E. F.

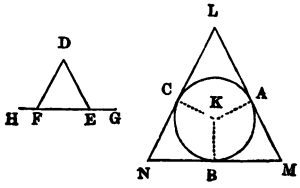
PROP. III.— PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.

(References—Prop. I. 13, 23, 32; III. 1, 17, 18.)

Let ABC be the given circle, and DEF the given triangle.

It is required to describe a triangle about the circle ABC equiangular to the triangle DEF.



CONSTRUCTION

Produce EF both ways to the points G, H;

find the centre K of the circle ABC, (III. 1) and from it draw any straight line KB;

at the point K, in the straight line KB, make the angle BKA equal to the angle DEG, (I. 23)

and the angle BKC equal to the angle DFH;

and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching the circle ABC. (III. 17.)

Then LMN shall be the triangle required.

DEMONSTRATION

Because LM, MN, NL, touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC,

the angles at the points A, B, C, *are right angles*; (III. 18)

and because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles;

and that two of them, KAM, KBM, are right angles,

the other two AKB, AMB, *are equal to two right angles*; (ax. 3)

but the angles DEG, DEF, are likewise equal to two right angles; (I. 13)

therefore the angles AKB, AMB, *are equal to the angles* DEG, DEF, (ax. 1)

of which AKB *is* equal to DEG; (constr.)

wherefore the remaining angle AMB, is equal to the remaining angle *DEF*. (ax. 3.)

In like manner, it may be demonstrated that the angle LNM is equal to DFE;

and therefore the remaining angle MLN *is equal to the remaining angle* EDF; (I. 32, and ax. 3)

wherefore the triangle LMN is equiangular to the triangle DEF;

and it is described about the circle ABC. Q. E. F.

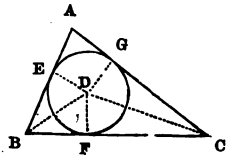
PROP. IV.— PROBLEM.

To inscribe a circle in a given triangle.

(References—Prop. I. 9, 12, 26; III. 16.)

Let the given triangle be ABC.

It is required to inscribe a circle in ABC.



CONSTRUCTION

Bisect the angles ABC, BCA, by the straight lines BD, CD, meeting one another in the point D, (I. 9)

from which draw DE, DF, DG, perpendiculars to AB, BC, CA; (I. 12)

from the centre D, with either of these distances, describe the circle EFG.

Then the circle EFG shall be inscribed in the triangle ABC.

DEMONSTRATION

Because the angle EBD is equal to the angle FBD, for the angle ABC is bisected by BD,

and that the right angle BED is equal to the right angle BFD; (ax. 11)

then, in the triangles EBD, FBD,

the two angles of the one are equal to two angles of the other, each to each;

and the *side* BD, which is opposite to one of the equal angles in each, *is common* to both;

therefore their other sides are equal; (I. 26)

wherefore DE *is equal to* DF;

for the same reason, DG is equal to DF;

therefore DE is equal to DG; (AX. 1)

wherefore the three straight lines DE, DF, DG, *are equal to one another;*

and the circle described from the centre D, at the distance of any of them, will pass through the extremities of the other two.

And because the angles at the points E, F, G, are right angles,

and the straight line which is drawn from the extremity of a diameter at right angles to it, touches the circle; (III. 16)

therefore the straight lines AB, BC, CA, *do each of them touch the circle;*

and therefore the circle EFG is inscribed in the triangle ABC. Q. E. F.

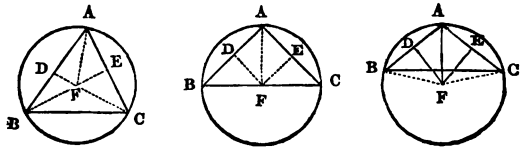
PROP. V.—PROBLEM.

To describe a circle about a given triangle.

(References—Prop. I. 4, 10, 11; III. 31.)

Let the given triangle be ABC.

It is required to describe a circle about ABC.



CONSTRUCTION

Bisect AB, AC, in the points D, E, (I. 10)

and from these points draw DF, EF, at right angles to AB, AC. (I. 11.)

DF, EF, produced meet one another;

for, if they do not meet, they are parallel,

wherefore AB, AC, *which are at right angles to them, are parallel;*

which is absurd;

let them meet in F, and join FA;

also if the point F be not in BC, join BF, CF;
and from the centre F, at the distance of BF, CF, or AF, describe the circle ABC.

Then the circle ABC shall be described about the triangle ABC.

DEMONSTRATION

Because AD is equal to DB, and DF common, and at right angles to AB,

therefore the base AF *is equal to the base* FB. (I. 4.)

In like manner, it may be shown that CF is equal to FA;
therefore BF is equal to FC; (ax. 1)

wherefore FA, FB, FC, *are equal to one another,*
and the circle described from the centre F, *at the distance of one of them, shall pass through the extremities of the other two.*

Therefore the circle ABC is described about the triangle ABC. Q. E. F.

Cor. And it is manifest that, when the centre of the circle falls within the triangle, *each of its angles is less than a right angle,* each of them being in a segment greater than a semicircle; (III. 31)

but when the centre is in one of the sides of the triangle, *the angle opposite to this side is a right angle,* being in a semicircle;

and if the centre fall without the triangle, *the angle opposite to the side beyond which it is, is greater than a right angle,* being in a segment less than a semicircle.

Wherefore, conversely, if the given triangle be acute-angled, *the centre of the circle falls within it;*

if it be a right-angled triangle, *the centre is in the side opposite to the right angle;*

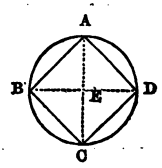
and, if it be an obtuse-angled triangle, *the centre falls without the triangle,* beyond the side opposite to the obtuse angle.

PROP. VI. — PROBLEM.

To inscribe a square in a given circle.

(References — Prop. I. 4, 11; III. I. 31.)

Let ABCD be the given circle.
It is required to inscribe a square in ABCD.



CONSTRUCTION

Draw the diameters AC, BD, at right angles to one another; (III. 1, and I. 11)

and join AB, BC, CD, DA.

Then the figure ABCD shall be inscribed in the circle ABCD.

DEMONSTRATION

Because BE is equal to ED, for E is the centre,

and that EA is common, and at right angles to BD;

the base BA *is equal to the base* AD; (I. 4)

and for the same reason, BC, CD, are each of them equal to BA or AD;

therefore the quadrilateral figure ABCD *is equilateral.*

It is also rectangular;

for the straight line BD, being the diameter of the circle ABCD, BAD is a semicircle;

wherefore the angle BAD *is a right angle;* (III. 31)

for the same reason, each of the angles ABC, BCD, CDA, is a right angle;

therefore the quadrilateral figure ABCD *is rectangular,*

and it has been shown to be equilateral;

therefore the quadrilateral figure ABCD *is a square;* (I. def. 30)

and it touches the circumferences of the circle at each of its angles.

Therefore the square ABCD is inscribed in the circle ABCD. Q. E. F.

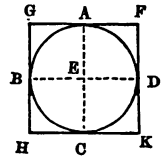
PROP. VII.—PROBLEM.

To describe a square about a given circle.

(References—Prop. I. 28, 34; III. 17, 18.)

Let ABCD **be the given circle.**

It is required to describe a square about it.



CONSTRUCTION

Draw two diameters, AC, BD, of the circle ABCD at right angles to one another;

and through the points A, B, C, D, draw FG, GH, HK, KF, touching the circle; (III. 17)

Then the figure GHKF, shall be described about the circle ABCD.

DEMONSTRATION

Because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A,

the angles at A *are right angles;* (III. 18)

for the same reason,

the angles at the points B, C, D, *are right angles;*

and because the angle AEB is a right angle, as likewise is EBG,

therefore GH *is parallel to* AC; (I. 28)

for the same reason,

AC *is parallel to* FK;

and in like manner it may be demonstrated that GF, HK, are each of them parallel to BED;

therefore the figures GK, GC, AK, FB, BK, are parallelograms;

and therefore GF *is equal to* HK, *and* GH *to* FK; (I. 34)

and because AC is equal to BD, and that AC is equal to each of the two GH, FK;

and BD to each of the two GF, HK;

therefore GH, FK, are each of them equal to GF, or HK;

wherefore the quadrilateral figure GHKF *is equilateral.*

It is also rectangular;

for GBEA being a parallelogram, and AEB a right angle,

therefore AGB *is likewise a right angle;* (I. 34)

in the same manner, it may be shown that the angles at H, K, F, are *right angles;*

therefore the quadrilateral figure GHKF *is rectangular;*

and it was demonstrated to be equilateral;

therefore GHKF *is a square;*

and all its sides touch the circumference of the circle;

wherefore the square GHKF is described about the circle ABCD. Q. E. F.

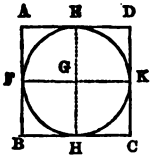
PROP. VIII.—PROBLEM.

To inscribe a circle in a given square.

(References—Prop. I. 10, 29, 31, 34; III. 16, cor.)

Let ABCD be the given square.

It is required to inscribe a circle in ABCD.



CONSTRUCTION

Bisect each of the sides AB, AD, in the points F, E; (I. 10)

and through E draw EH parallel to AB or DC; (I. 31)

and through F draw FK parallel to AD or BC, cutting EH in G;

from the centre G, at the distance GE, GF, or GK, describe the circle EFHK.

Then the circle EFHK shall be inscribed in the square ABCD.

DEMONSTRATION

Because each of the figures AK, KB, AH, HD, AG, GC, BG, GD, is a parallelogram (constr.)

their opposite sides are equal; (I. 34)

and because AD is equal to AB, (I. def. 30) and that AE is the half of AD, and AF the half of AB,

therefore AE *is equal to* AF; (ax. 7)

and the sides opposite to these are equal,

wherefore FG *is equal to* GE;

in the same manner it may be demonstrated that GH, GK, are each of them equal to FG or GE;

therefore the four straight lines GE, GF, GH, GK, *are equal to one another;*

and the circle described from the centre G, *at the distance of one of them, shall pass through the extremities of the other three.*

And because the angles at the points E, F, H, K, are right angles; (I. 29)

and that the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle; (III. 16, Cor.)

therefore each of the straight lines AB, BC, CD, DA, *touches the circle* EFHK,

wherefore the circle EFHK is inscribed in the square ABCD. Q. E. F.

PROP. IX.—Problem.

To describe a circle about a given square.

(References — Prop. I. 6, 8.)

Let ABCD be the given square.

It is required to describe a circle about ABCD.

CONSTRUCTION

Join AC, BD, cutting one another in E;

then, from the centre E, at the distance of EA, EB, EC, or ED, describe the circle ABCD.

Then the circle ABCD shall be described about the square ABCD.

DEMONSTRATION

Because, in the two triangles, DAC, BAC, DA is equal to AB (I. def. 30), and AC common,

the two sides DA, AC, are equal to the two BA, AC, each to each;

and the base DC is equal to the base BC;

therefore the angle DAC *is equal to the angle* BAC; (I. 8)

and the angle DAB *is bisected by the straight line* AC.

In the same manner it may be demonstrated that the angle ABC, BCD, CDA, are severally bisected by the straight lines BD, AC.

Hence, because the angle DAB is equal to the angle ABC, (I. def. 30) and that the angle EAB is the half of DAB, and EBA the half of ABC;

the angle EAB is equal to the angle EBA; (ax. 7)

therefore the side EA *is equal to the side* EB. (I. 6.)

In the same manner it may be demonstrated that the straight lines EC, ED, are each of them equal to EA or EB;

therefore the four straight lines EA, EB, EC, ED, *are equal to one another;*

and the circle described from the centre E, at the distance of one of them, shall pass the extremities of the other three;

wherefore the circle ABCD is described about the square ABCD. Q. E. F.

PROP. X.—PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

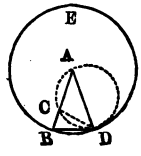
(References—Prop. I. 5, 6, 32; II. 11; III. 32, 37; IV. 1, 5.)

CONSTRUCTION

Take any straight line AB, and divide it in the point C, so that the rectangle AB, BC, may be equal to the square of CA; (II. 11)

and from the centre A, at the distance AB, describe the circle BDE, in which place the straight line BD equal to AC, which is not greater than the diameter of the circle BDE; (IV. 1) and join DA.

Then the triangle ABD shall be such as is required, that is, each of the angles ABD, ADB shall be double of the angle BAD.



Join DC, and about the triangle ADC describe the circle ACD. (IV. 5.)

DEMONSTRATION

Because the rectangle AB, BC, is equal to the square of AC, and that AC is equal to BD, (constr.)

the rectangle AB, BC, *is equal to the square of* BD; (AX. 1)

and because from the point B without the circle ACD, two straight lines BCA, BD, are drawn to the circumferences, one of which cuts, and the other meets the circle,

and that the rectangle AB, BC, contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD, which meets it;

the straight line BD *touches the circle* ACD; (III. 37)

and because BD touches the circle, and DC is drawn from the point of contact D,

the angle BDC *is equal to the angle* DAC *in the alternate segment of the circle*; (III. 32)

to each of these add the angle CDA;

therefore the whole angle BDA *is equal to the two angles* CDA, DAC; (AX. 2)

but the exterior angle BCD is equal to the angles CDA, DAC; (I. 32)

therefore also BDA *is equal to* BCD; (AX. 1)

but BDA is equal to the angle CBD, (I. 5) because the side AD is equal to the side AB;

therefore CBD, or DBA, is equal to BCD; (AX. 1)

and consequently the three angles BDA, DBA, BCD, *are equal to one another.*

Again, because the angle DBC is equal to the angle BCD,

the side DB *is equal to the side* DC; (I. 6)

but BD was made equal to CA;

therefore also CA *is equal to* CD; (AX. 1)

and the angle CDA is equal to the angle DAC; (I. 5)

therefore the angles CDA, DAC, *together are double of the angle* DAC;

but BCD is equal to the angles CDA, DAC; (I. 32)

therefore also BCD *is double of* DAC;

and BCD *was proved* to be equal to each of the angles BDA, DBA,

therefore each of the angles BDA, DBA, is double of the angle DAB.

Wherefore an isosceles triangle ABD has been described, having each of angles at the base double of the third angle. Q. E. F.

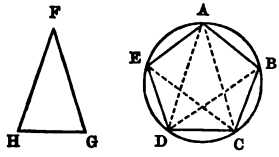
PROP. XI.—PROBLEM.

To inscribe an equilateral and equiangular pentagon in a given circle.

(References — Prop. I. 9; III. 26, 27, 29; IV. 2, 10.)

Let ABCDE be the given circle.

It is required to inscribe an equilateral and equiangular pentagon in the circle ABCDE.



CONSTRUCTION

Describe an isosceles triangle FGH, having each of the angles at G, H, double of the angle at F; (IV. 10)

and in the circle ABCDE inscribe the triangle ACD, equiangular to the triangle FGH, (IV. 2)

so that the angle CAD may be equal to the angle at F, and each of the angles ACD, CDA, equal to the angle at G or H;

wherefore each of the angles ACD, CDA, *is double of the angle* CAD;

bisect the angles ACD, CDA, by the straight lines CE, DB; (I. 9) and join AB, BC, DE, EA.

Then ABCDE shall be the equilateral and equiangular pentagon required.

DEMONSTRATION

Because each of the angles ACD, CDA, is double of CAD, and that they are bisected by the straight lines CE, DB,

the five angles DAC, ACE, ECD, CDB, BDA, *are equal to one another;*

but equal angles stand upon equal circumferences; (III. 26)

therefore the five circumferences AB, BC, CD, DE, EA, *are equal to one another*

and equal circumferences are subtended by equal straight lines; (III. 29)

therefore the five straight lines AB, BC, CD, DE, EA, *are equal to one another;*

wherefore the pentagon ABCDE is equilateral.

It is also equiangular;

for, because the circumference AB is equal to the circumference DE, if to each be added BCD,

the whole ABCD *is equal to the whole* EDCB; (ax. 2)

but the angle AED stands on the circumference ABCD, and the angle BAE on the circumference EDCB;

therefore the angle BAE *is equal to the angle* AED; (III. 27)

for the same reason,

each of the angles ABC, BCD, CDE, *is equal to the angle* BAE *or* AED;

therefore the pentagon ABCDE is equiangular.

And it has been shown that it is equilateral;

wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed. Q. E. F.

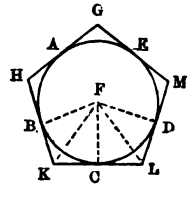
PROP. XII.—Problem.

To describe an equilateral and equiangular pentagon about a given circle.

(References — Prop. I. 4, 8, 26, 47; III. 17, 18, 27; IV. 11.)

Let ABCDE be the given circle.

It is required to describe an equilateral and equiangular pentagon about the circle ABCDE.



CONSTRUCTION

Let the angular points of a pentagon, inscribed in the circle, by the last proposition, be in the points A, B, C, D, E, so that the circumferences AB, BC, CD, DE, EA, are equal; (IV. 11)

and through the points A, B, C, D, E, draw GH, HK, KL, LM, MG, touching the circle. (III. 17.)

Then the figure GHKLM shall be the equilateral and equiangular pentagon required.

Take the centre F, and join FB, FK, FC, FL, FD.

DEMONSTRATION

Because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F,
FC is perpendicular to KL; (III. 18)

therefore each of the angles at C *is a right angle;*

for the same reason,

the angles at the points B, D, *are right angles;*

and because FCK is a right angle,

the square of FK *is equal to the squares of* FC, CK; (I. 47)

for the same reason, the square of FK is equal to the squares of FB, BK;

therefore the squares of FC, CK, *are equal to the squares of* FB, BK; (ax. 1)

of which the sqnare of FC is equal to the square of FB;
therefore the remaining square of CK is equal to the remaining square of BK; (ax. 3)

and the straight line CK *equal to* BK.

Hence in the two triangles BFK, CFK,
because FB is equal to FC, and FK common to both,
the two BF, FK, are equal to the two CF, FK, each to each;
and the base BK was proved equal to the base KC;
therefore the angle BFK is equal to the angle KFC, (I. 8) and the angle BKF to FKC; (I. 4)

wherefore the angle BFC *is double of the angle* KFC, *and* BKC *double of* FKC;

for the same reason,

the angle CFD *is double of the angle* CFL, *and* CLD *double of* CLF;

and because the circumference BC is equal to the circumference CD,

the angle BFC *is equal to the angle* CFD; (III. 27)

and BFC is double of the angle KFC, and CFD double of CFL;

therefore the angle KFC *is equal to the angle* CFL, (ax. 7)

and the right angle FCK is equal to the right angle FCL.

Hence in the two triangles FKC, FLC,

there are two angles of one equal to two angles of the other, each to each,

and the side FC, which is adjacent to the equal angles in each, is common to both ;

therefore the other sides shall be equal to the other sides, and the third angle to the third angle ; (I. 26)

therefore the straight line KC *is equal to* CL, *and the angle* FKC *to the angle* FLC ;

and because KC is equal to CL,

therefore KL *is double of* KC ;

in the same manner it may be shown that

HK *is double of* BK.

And because BK is equal to KC, as was demonstrated, and that KL is double of KC, and HK double of BK,

therefore HK *is equal to* KL ; (ax. 6)

in like manner, it may be shown

that GH, GM, ML, *are each of them equal to* HK or KL ;

therefore the pentagon GHKLM is equilateral.

It is also equiangular ;

for, since the angle FKC is equal to the angle FLC,

and the angle HKL double of the angle FKC, and KLM double of FLC, as was before demonstrated,

the angle HKL *is equal to* KLM ; (ax. 6)

and in like manner, it may be shown that each of the angles KHG, HGM, GML, is equal to the angle HKL or KLM ;

therefore the five angles GHK, HKL, KLM, LMG, MGH, *are equal to one another,*

therefore the pentagon GHKLM is equiangular.

and it is equilateral, as was demonstrated ;

and it is described about the circle ABCDE. Q. E. F.

PROP. XIII.—Problem.

To inscribe a circle in a given equilateral and equiangular pentagon.

(References—Prop. I. 4, 9, 12, 26 ; III. 16.)

Let ABCDE be the given equilateral and equiangular pentagon.

It is required to inscribe a circle in the pentagon ABCDE.

DEMONSTRATION

It may be demonstrated, as in the same manner as in the preceding proposition, that

the angles CBA, BAE, AED, *are bisected by the straight lines* FB, FA, FE;

and because the angle BCD is equal to the angle CDE,

and that FCD is the half of the angle BCD, and CDF the half of CDE;

the angle FCD is equal to FDC; (ax. 7)

wherefore the side CF *is equal to the side* FD. (I. 6.)

In like manner it may be demonstrated that FB, FA, FE, are each of them equal to FC or FD;

therefore the five straight lines FA, FB, FC, FD, FE, are equal to one another;

and the circle described from the centre F, *at the distance of one of them, shall pass through the extremities of the other four,*

wherefore the circle ABCDE is described about the equilateral and equiangular pentagon ABCDE. Q. E. F.

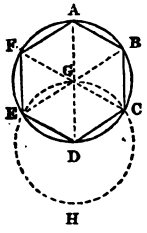
PROP. XV.—PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

(References — Prop. I. 5 cor., 13, 15, 32; III. 1, 26, 27, 29.)

Let ABCDEF be the given circle.

It is required to inscribe an equilateral and equiangular hexagon in it.



CONSTRUCTION

Find the centre G of the circle ABCDEF, (III. 1) and draw the *diameter* AGD;

and from D as a centre, at the distance DG, describe the circle EGCH;

join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA.

Then ABCDEF shall be an equilateral and equiangular hexagon inscribed as required in the given circle.

DEMONSTRATION

Because G is the centre of the circle ABCDEF,

therefore GE *is equal to* GD;

and because D is the centre of the circle EGCH,

therefore DE *is equal to* DG;

wherefore GE is equal to ED, (ax. 1)

and the triangle EGD *is equilateral;*

and therefore its three angles EGD, GDE, DEG, are equal to one another; (I. 5 cor.)

but the three angles of a triangle are equal to two right angles; (I. 32)

therefore the angle EGD *is the third part of two right angles;*

in the same manner it may be demonstrated that

the angle DGC *is also the third part of two right angles;*

and because the straight line GC makes with EB the adjacent angles, EGC, CGB, equal to two right angles, (I. 13)

the remaining angle CGB is the third part of two right angles;

therefore the angles EGD, DGC, CGB, *are equal to one another;*

and to these are equal the vertical opposite angles BGA, AGF, FGE; (I. 15)

therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE, *are equal to one another;*

but equal angles stand upon equal circumferences; (III. 26)

therefore the six circumferences AB, BC, CD, DE, EF, FA, *are equal to one another;*

and equal circumferences are subtended by equal straight lines; (III. 29)

therefore the six straight lines are equal to one another,

and the hexagon ABCDEF is equilateral.

It is also equiangular;

for, since the circumference AF is equal to ED,

to each of these add the circumference ABCD;

therefore the whole circumference FABCD is equal to the whole *EDCBA;*

and the angle FED stands upon the circumference FABCD, and the angle AFE upon EDCBA;

therefore the angle AFE *is equal to* FED; (III. 27)

in the same manner it may be demonstrated that

the other angles of the hexagon ABCDEF *are each of them equal to the angle* AFE *or* FED;

therefore the hexagon ABCDEF is equiangular;

and it is equilateral, as was shown;

and it is inscribed in the given circle ABCDEF. Q. E. F.

Cor. From this it is manifest that the side of the hexagon is equal to the straight line from the centre, that is, to the semidiameter of the circle.

And if through the points A, B, C, D, E, F, there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be *described about it*, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be *inscribed* in a given equilateral and equiangular hexagon, and *circumscribed about it*, by a method like that used for the pentagon.

PROP. XVI.—PROBLEM.

To inscribe an equilateral and equiangular quindecagon in a given circle.

(References—Prop. III. 27, 30; IV. 1, 2, 11.)

Let ABCD be the given circle.

It is required to inscribe an equilateral and equiangular quindecagon in it.

CONSTRUCTION

Let AC be the side of an equilateral triangle inscribed in the circle ABCD, (IV. 2)

and AB the side of an equilateral and equiangular pentagon inscribed *in the same*; (IV. 11)

bisect the circumference BC in E, (III. 30) and join BE, EC;
draw straight lines equal to these, and place them contiguous to each other round the circle. (IV. 1.)

Then ABCDF shall be an equilateral and equiangular quindecagon described as required in the circle ABCD.

DEMONSTRATION

Because of such equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, which is the third part of the whole, contains five;
and the circumference AB, which is the fifth part of the whole, contains three;
therefore their difference BC *contains two such parts,*
but BC was bisected in E; (constr.) therefore BE, EC are each of them the fifteenth part of the whole circumference ABCD;
wherefore the figure ABECDF is equilateral.

And because each of its angles stands upon thirteen-fifteenths of the circumference,
therefore the figure ABECDF is equiangular. (III. 27.)
Wherefore an equilateral and equiangular quindecagon has been inscribed in the circle ABC. Q. E. F.

And in the same manner as was done in the pentagon, if through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be *described about* it; and likewise, as in the pentagon, a circle may be *inscribed in* a given equilateral and equiangular quindecagon, and *circumscribed about* it.

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